

Maximal Subalgebras for Lie Superalgebras of Cartan Type

Wei Bai¹

*Department of Mathematics, Harbin Institute of Technology,
Harbin 150006, P. R. China*

*School of Mathematical Sciences, Harbin Normal University,
Harbin 150025, P. R. China*

Wende Liu^{*2}

*Department of Mathematics, Harbin Institute of Technology,
Harbin 150006, P. R. China*

Xuan Liu

*Applied Mathematics Department, The University of Western Ontario,
London, N6A 5B7, Canada*

Hayk Melikyan³

*Department of Mathematics and Computer Science, North Carolina Central University
Durham, NC 27713, USA*

Abstract

The maximal graded subalgebras for four families of Lie superalgebras of Cartan type over a field of prime characteristic are studied. All maximal reducible graded subalgebras are described completely and their isomorphism classes, dimension formulas are found. The classification of maximal irreducible graded subalgebras is reduced to the classification of the maximal irreducible subalgebras for the classical Lie superalgebras $\mathfrak{gl}(m, n)$, $\mathfrak{sl}(m, n)$ and $\mathfrak{osp}(m, n)$.

Keywords: Lie superalgebras; maximal graded subalgebras
Mathematics Subject Classification 2010: 17B50, 17B05.

0. Introduction

Since V. G. Kac [1] classified the finite dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero, the theory of Lie superalgebras has undergone a significant development (for example [2, 3]). Over a field of finite characteristic, however, the classification problem is still open for the finite dimensional simple Lie superalgebras [4, 5]. Even recently, new simple Lie superalgebras over a field of characteristic $p = 3$ were constructed [5, 6].

^{*}Corresponding author: wendeliu@ustc.edu.cn (W. Liu)

¹Supported by NSF of the Education Department of HLJP (12521158)

²Supported by NSF of China (11171055) and NSF of HLJP (JC201004, A2010-03)

³Supported by part NSF Grant # 0833184

In general, study of the maximal subsystems of an algebraic system, such as finite groups, Lie groups, Lie (super)algebras, is an essential part of structural characterization of the system. In classical Lie theory, the classification of maximal subalgebras of simple Lie algebras over the field of complex numbers is one of the beautiful results of that theory which was due to E. Dynkin [7, 8]. In classical modular Lie theory there is a series of papers by G. Seitz and his students devoted to the study of the maximal subgroups of simple algebraic groups over fields of positive characteristic. These investigations were summarized by G. Seitz in his two publications [9, 10] which generalize E. Dynkin's classification of the maximal subgroups of simple Lie groups over the field of complex numbers [8] to simple algebraic groups over fields of characteristic $p > 7$. The study of maximal subalgebras of different classes of (super)algebras has been the focus of several researchers. The maximal subalgebras of Jordan (super)algebras were studied by M. Racine [11, 12], A. Elduque, J. Laliena and S. Sacristan [13, 14]. The maximal graded subalgebras of affine Kac-Moody algebras were classified in [15]. The fourth author of the present paper summarized his investigations on maximal subalgebras in Cartan type simple Lie algebras over the field of characteristic $p > 3$ in his paper [16].

Let L be a finite dimensional simple Lie superalgebras of Cartan type W , S , H or K with a \mathbb{Z} -grading $L = \bigoplus_{i \geq -2} L_i$. The present paper is devoted to characterizing the maximal graded subalgebras of L . To this end, we construct a series of graded subalgebras of L and state the necessary and sufficient conditions for their maximality. Moreover, the number of isomorphism classes and the dimension formulas of all maximal graded subalgebras are completely determined except for maximal irreducible graded subalgebras. Note that the null of L is isomorphic to a classical Lie superalgebra (see Lemma 2.1(3)). Thus the classification of the maximal irreducible graded subalgebras of L is reduced to that of the maximal irreducible subalgebras of a classical Lie superalgebra. Moreover, we give necessary and sufficient conditions for the existence of maximal irreducible graded subalgebras of L . We should mention that the present work which partially generalizes the results of [16] is motivated by a paper by A. I. Kostrikin and I. R. Shafarevich [17] on the structure theory of modular Lie algebras.

We close this introduction by establishing the following conventions: The underlying field \mathbb{F} is an algebraically closed field of characteristic $p > 3$. In addition to the standard notation \mathbb{Z} , we write \mathbb{N} for the set of nonnegative integers. The field of two elements is denoted by $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. For a proposition P , put $\delta_P = 1$ if P is true and $\delta_P = 0$ otherwise. *All subspaces, subalgebras and submodules are assumed to be \mathbb{Z}_2 -graded and all the homomorphisms of \mathbb{Z} -graded superalgebras are both \mathbb{Z}_2 -homogeneous and \mathbb{Z} -homogeneous.*

1. Basics

Fix two positive integers $m, n \in \mathbb{N} \setminus \{1\}$. Put

$$\mathbf{I}_0 = \overline{1, m}, \mathbf{I}_1 = \overline{m+1, m+n}, \mathbf{I} = \mathbf{I}_0 \cup \mathbf{I}_1,$$

where $\overline{k, s} = \{k, k+1, \dots, s\}$ with the convention $\overline{k, s} = \emptyset$ whenever $k > s$. Write

$$\mathbf{A}(m) = \{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m \mid 0 \leq \alpha_i \leq p-1, i \in \mathbf{I}_0\}.$$

Let $\mathcal{O}(m)$ be the *divided power algebra* with \mathbb{F} -basis $\{x^{(\alpha)} \mid \alpha \in \mathbf{A}(m)\}$ and $\Lambda(n)$ be the *exterior superalgebra* of n variables $x_{m+1}, x_{m+2}, \dots, x_{m+n}$. The tensor product

$$\mathcal{O}(m, n) = \mathcal{O}(m) \otimes \Lambda(n)$$

is an associative superalgebra with respect to the usual \mathbb{Z}_2 -grading. Let

$$\mathbf{B}(n) = \{\langle i_1, i_2, \dots, i_k \rangle \mid 0 \leq k \leq n; m+1 \leq i_1 < i_2 < \dots < i_k \leq m+n\}$$

be the set of k -tuples of strictly increasing integers in \mathbf{I}_1 , $0 \leq k \leq n$. For $u = \langle i_1, i_2, \dots, i_k \rangle \in \mathbf{B}(n)$, write $x^u = x_{i_1} x_{i_2} \dots x_{i_k}$ ($x^\emptyset = 1$). If $g \in \mathcal{O}(m)$ and $f \in \Lambda(n)$, we write gf instead of $g \otimes f$. Then $\mathcal{O}(m, n)$ has a \mathbb{Z}_2 -homogeneous \mathbb{F} -basis

$$\{x^{(\alpha)} x^u \mid \alpha \in \mathbf{A}(m), u \in \mathbf{B}(n)\}.$$

For $i \in \mathbf{I}_0$ and $\varepsilon_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{im})$, write x_i for $x^{(\varepsilon_i)}$. Let $\partial_1, \dots, \partial_{m+n}$ be the *superderivations* of the superalgebra $\mathcal{O}(m, n)$ such that $\partial_i(x_j) = \delta_{i=j}$. The *parity* of ∂_i is $|\partial_i| = \bar{0}$ if $i \in \mathbf{I}_0$ and $\bar{1}$ if $i \in \mathbf{I}_1$. Hereafter the symbol $|x|$ implies that x is a \mathbb{Z}_2 -homogeneous element. Put

$$W(m, n) = \text{span}_{\mathbb{F}} \{a \partial_i \mid a \in \mathcal{O}(m, n), i \in \mathbf{I}\},$$

which is a finite dimensional simple Lie superalgebra, called *Witt superalgebra*. Consider the linear mapping called divergence:

$$\text{div} : W(m, n) \longrightarrow \mathcal{O}(m, n), \quad \text{div}(f \partial_i) = (-1)^{|f||\partial_i|} \partial_i(f).$$

Set $S(m, n) = [\bar{S}(m, n), \bar{S}(m, n)]$, where $\bar{S}(m, n) = \ker(\text{div})$. Then we have

$$S(m, n) = \text{span}_{\mathbb{F}} \{D_{ij}(a) \mid a \in \mathcal{O}(m, n), i, j \in \mathbf{I}\},$$

where

$$D_{ij}(a) = (-1)^{|\partial_i||\partial_j|} \partial_i(a) \partial_j - (-1)^{(|\partial_i|+|\partial_j|)|a|} \partial_j(a) \partial_i \quad \text{for } a \in \mathcal{O}(m, n).$$

$S(m, n)$ is a simple Lie superalgebra, called *special superalgebra*.

For $j \in \{1, \dots, 2\lfloor \frac{m}{2} \rfloor, m+1, \dots, m+n\}$, we put

$$\sigma(j) = \begin{cases} -1, & j \in \overline{\lfloor \frac{m}{2} \rfloor + 1, 2\lfloor \frac{m}{2} \rfloor}; \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad j' = \begin{cases} j + \lfloor \frac{m}{2} \rfloor, & j \in \overline{1, \lfloor \frac{m}{2} \rfloor}; \\ j - \lfloor \frac{m}{2} \rfloor, & j \in \overline{\lfloor \frac{m}{2} \rfloor + 1, 2\lfloor \frac{m}{2} \rfloor}; \\ j, & \text{otherwise.} \end{cases}$$

Suppose $m = 2r$ is even. Define an even linear mapping $D_H : \mathcal{O}(m, n) \longrightarrow W(m, n)$ by $D_H(a) = \sum_{i \in \mathbf{I}} \sigma(i) (-1)^{|\partial_i||a|} \partial_i(a) \partial_{i'}$. Put

$$\bar{H}(m, n) = \text{span}_{\mathbb{F}} \{D_H(a) \mid a \in \mathcal{O}(m, n)\}.$$

Write $\bar{\mathcal{O}}(m, n)$ for the quotient superspace $\mathcal{O}(m, n)/\mathbb{F} \cdot 1$. We can view D_H as a linear operator of $\bar{\mathcal{O}}(m, n)$ since the kernel of D_H is $\mathbb{F} \cdot 1$. Thus we have $\bar{H}(m, n) \cong (\bar{\mathcal{O}}(m, n), [\ , \])$, where the bracket is:

$$[a, b] = D_H(a)(b) \quad \text{for } a, b \in \bar{\mathcal{O}}(m, n).$$

Its derived algebra $H(m, n)$ is simple, called *Hamiltonian Lie superalgebra*.

Suppose $m = 2r + 1$ is odd. Define an even linear mapping $D_K : \mathcal{O}(m, n) \rightarrow W(m, n)$ by $D_K(a) = D_H(a) + \partial_m(a)\mathfrak{D} + \Delta(a)\partial_m$, where $\mathfrak{D} = \sum_{i \in \mathbf{I} \setminus \{m\}} x_i \partial_i$ and $\Delta(a) = 2a - \mathfrak{D}(a)$. Put

$$\overline{K}(m, n) = \text{span}_{\mathbb{F}}\{D_K(a) \mid a \in \mathcal{O}(m, n)\}.$$

Since D_K is injective, we have $\overline{K}(m, n) \cong (\mathcal{O}(m, n), [\ , \])$, where the bracket is:

$$[a, b] = D_H(a)b + \Delta(a)\partial_m(b) - \partial_m(a)\Delta(b) \quad \text{for } a, b \in \mathcal{O}(m, n).$$

Its derived algebra $K(m, n)$ is simple, called *contact Lie superalgebra*.

For simplicity, hereafter, we write X for $X(m, n)$, where $X = \mathcal{O}, \overline{\mathcal{O}}, W, \overline{S}, S, \overline{H}, H, \overline{K}$ or K . Let us consider the standard \mathbb{Z} -grading of L , where $L = \mathcal{O}, W, S, \overline{H}, H$ or K . Define the \mathbb{Z} -degrees of x_i and ∂_i to be $\text{zd}(x_i) = -\text{zd}(\partial_i) = 1 + \delta_{L=K}\delta_{i=m}$, $i \in \mathbf{I}$. Hereafter, the symbol $\text{zd}(x)$ always implies that x is a \mathbb{Z} -homogeneous element. Put $\xi = (m + \delta_{L=K})(p - 1) + n$. Then we have:

$$\begin{aligned} \mathcal{O} &= \bigoplus_{i=-1}^{\xi} \mathcal{O}_i, \quad \mathcal{O}_i = \text{span}_{\mathbb{F}}\{f \in \mathcal{O} \mid \text{zd}(f) = i\}; \\ W &= \bigoplus_{i=-1}^{\xi-1} W_i, \quad W_i = \text{span}_{\mathbb{F}}\{f \partial_j \mid f \in \mathcal{O}_{i+1}, j \in \mathbf{I}\}; \\ S &= \bigoplus_{i=-1}^{\xi-2} S_i, \quad S_i = \text{span}_{\mathbb{F}}\{D_{jk}(f) \in W \mid f \in \mathcal{O}_{i+2}, j, k \in \mathbf{I}\}; \\ \overline{H} &= \bigoplus_{i=-1}^{\xi-2} \overline{H}_i, \quad \overline{H}_i = \text{span}_{\mathbb{F}}\{f \mid f \in \mathcal{O}_{i+2}\}; \quad H = \bigoplus_{i=-1}^{\xi-3} H_i; \\ K &= \bigoplus_{i=-2}^{\xi-2-\delta_{n-m-3 \equiv 0 \pmod{p}}} K_i, \quad K_i = \text{span}_{\mathbb{F}}\{f \mid f \in \mathcal{O}_{i+2}\}. \end{aligned}$$

We adopt the following conventions:

- (1) $L = H$ implies that $m = 2r$ is even; $L = K$ implies that $m = 2r + 1$ is odd.
- (2) K can be viewed as a \mathbb{Z} -graded subalgebra of W when $\text{zd}(x_m) = -\text{zd}(\partial_m) = 2$ for W . Thus, L is a \mathbb{Z} -graded subalgebra of W , where $L = S, H$ or K .
- (3) For $L = K$, we write z for x_m .
- (4) Write $\text{alg}(S)$ for the subalgebra of L generated by a subset S .

A proper subalgebra M of a \mathbb{Z} -graded Lie superalgebra L is called a *maximal graded subalgebra* (MGS) provided that M is \mathbb{Z} -graded and no nontrivial \mathbb{Z} -graded subalgebras of L strictly contains M . Since L_{-1} is an irreducible L_0 -module, it is clear that $\bigoplus_{i \geq 0} L_i$ is an MGS of L . Any other MGS, M , must satisfy exactly one of the following conditions:

- (I) $M_{-1} = L_{-1}$ and $M_0 = L_0$;
- (II) M_{-1} is a nontrivial proper subspace of L_{-1} ;
- (III) $M_{-1} = L_{-1}$ and $M_0 \neq L_0$.

Let \mathfrak{G}_0 be a subalgebra of L_0 . \mathfrak{G}_0 is called *reducible* (resp. *irreducible*) if the \mathfrak{G}_0 -module L_{-1} is reducible (resp. irreducible). An MGS $\mathfrak{G} = \sum_{i \geq -2} \mathfrak{G}_i$ of L is called *maximal reducible graded* (resp. *maximal irreducible graded*) if \mathfrak{G}_0 is reducible (resp. irreducible).

2. Preliminary Results

In order to simplify our considerations, in this section, we establish some technical lemmas. For $L = H$ or K , we redescribe L in an appropriate form and establish a suitable automorphism of L by virtue of a nondegenerate skew supersymmetric bilinear form on L_{-1} .

As in the case of Lie superalgebras of characteristic 0 [1] or modular Lie algebras [17, 18, 19], it is easy to show the following:

Lemma 2.1. *Let $L = W, S, H$ or K .*

- (1) L is transitive.
- (2) L is generated by its local part, $L = \text{alg}(L_{-1} + L_0 + L_1)$.
- (3) For the null of L , the following conclusions hold:

$$\begin{aligned} W(m, n)_0 &\cong \mathfrak{gl}(m, n); \quad S(m, n)_0 \cong \mathfrak{sl}(m, n); \\ H(2r, n)_0 &\cong \mathfrak{osp}(2r, n); \quad K(2r+1, n)_0 \cong \mathfrak{osp}(2r, n) \oplus \mathbb{F}I_{2r+n}. \end{aligned}$$

When $L = W$ or S , we know that L_{-1} is spanned by the standard ordered \mathbb{F} -basis

$$\{\partial_i \mid i \in \mathbf{I}\}. \quad (2.1)$$

For a \mathbb{Z}_2 -graded subspace $V = V_0 \oplus V_1$ of L_{-1} , the super-dimension is denoted by

$$\text{superdim} V = (\dim V_0, \dim V_1).$$

When $L = H$ or K , we redescribe L in a desired form. For $i \in \mathbb{N} \setminus \{0\}$, write A_i for an $i \times i$ matrix, and particularly, let I_i be the $i \times i$ unit matrix. Denote by \sqrt{a} a fixed solution of the equation $x^2 = a$ in \mathbb{F} , where $a = -1, 2$. Put

$$y_i = \begin{cases} x_i, & i \in \mathbf{I}_0 \cup \overline{m+2q+1, m+n}; \\ \frac{x_i + \sqrt{-1}x_{i+q}}{\sqrt{2}}, & i \in \overline{m+1, m+q}; \\ \frac{x_{i-q} - \sqrt{-1}x_i}{\sqrt{2}}, & i \in \overline{m+q+1, m+2q}, \end{cases}$$

where $0 \leq d \leq n$, $q = \lfloor \frac{n-d}{2} \rfloor$. Then there exists an invertible matrix A_{m+n} such that $(y_1, \dots, y_{m+n})A = (x_1, \dots, x_{m+n})$. Obviously, $|y_i| = |x_i|$ and $\text{zd}(y_i) = \text{zd}(x_i)$, $i \in \mathbf{I}$. By [20, Lemma 2.5], we have:

$$\{y^{(\alpha)}y^u \mid \alpha \in \mathbf{A}(m), u \in \mathbf{B}(n)\}$$

is an \mathbb{F} -basis of \mathcal{O} , where $y^{(\alpha)} = x^{(\alpha)}$ and $y^u = y_{i_1}y_{i_2} \cdots y_{i_k}$ when $u = \langle i_1, i_2, \dots, i_k \rangle$. The basis-element $y^{(\alpha)}y^u$ is called a *monomial*.

Write $(D_1, \dots, D_{m+n}) = (\partial_1, \dots, \partial_{m+n})A^t$. Then we have

$$D_i = \begin{cases} \partial_i, & i \in \mathbf{I}_0 \cup \overline{m+2q+1, m+n}; \\ \frac{\partial_i - \sqrt{-1}\partial_{i+q}}{\sqrt{2}}, & i \in \overline{m+1, m+q}; \\ \frac{\partial_{i-q} + \sqrt{-1}\partial_i}{\sqrt{2}}, & i \in \overline{m+q+1, m+2q}. \end{cases}$$

By a direct computation, we have:

$$D_i(y_j) = \delta_{i=j}, \quad \sum_{i \in \mathbf{I} \setminus \{2r+1\}} y_i D_i = \mathfrak{D}.$$

When $m = 2r$, define an even linear mapping $E_H : \mathcal{O} \rightarrow W$ by

$$E_H(a) = \sum_{i \in \mathbf{I}} \sigma(i) (-1)^{|D_i||a|} D_i(a) D_{\tilde{i}},$$

where

$$\tilde{i} = \begin{cases} i', & i \in \mathbf{I}_0 \cup \overline{m+2q+1, m+n}; \\ i+q, & i \in \overline{m+1, m+q}; \\ i-q, & i \in \overline{m+q+1, m+2q}. \end{cases}$$

When $m = 2r+1$, define an even linear mapping $E_K : \mathcal{O} \rightarrow W$ by

$$E_K(a) = E_H(a) + D_m(a) \mathfrak{D} + \Delta(a) D_m.$$

A direct computation shows that $D_L = E_L$. Note that L_{-1} is spanned by the standard ordered \mathbb{F} -basis

$$\{y_i \mid i \in \overline{1, 2r} \cup \overline{m+1, m+n}\}. \quad (2.2)$$

Define an even bilinear form $\beta : L_{-1} \times L_{-1} \rightarrow \mathbb{F}$ satisfying

$$\beta(u, v) = \sum_{i \in \mathbf{I}} \sigma(i) (-1)^{|D_i||u|} D_i(u) D_{\tilde{i}}(v) \quad \text{for } u, v \in L_{-1}.$$

Then the matrix of β in the ordered basis (2.2) is

$$J = \left(\begin{array}{c|c|ccc} 0 & I_r & & & \\ \hline -I_r & 0 & & & \\ \hline & & 0 & & \\ \hline & & 0 & -I_q & 0 \\ & & -I_q & 0 & 0 \\ & & 0 & 0 & -I_{n-2q} \end{array} \right). \quad (2.3)$$

Clearly, β is a nondegenerate skew supersymmetric bilinear form on L_{-1} .

An \mathbb{F} -basis of L_{-1} in which the matrix of β is J is called *generalized orthosymplectic*. Let $V = V_0 \oplus V_1$ be a subspace of L_{-1} . Suppose $2a$ (resp. d) is the rank of β restricted to V_0 (resp. V_1). A \mathbb{Z}_2 -homogeneous basis of V

$$\{e_1, \dots, e_a, e_{r+1}, \dots, e_{r+a}; e_{a+1}, \dots, e_b \mid e_{m+1}, \dots, e_{m+c}; e_{m+n-d+1}, \dots, e_{m+n}\} \quad (2.4)$$

is called a β -basis of V , if

$$\{e_1, \dots, e_a, e_{r+1}, \dots, e_{r+a}; e_{a+1}, \dots, e_b\}, \quad 0 \leq a \leq b \leq r$$

is an \mathbb{F} -basis of V_0 satisfying

$$\beta(e_i, e_j) = -\beta(e_j, e_i) = \begin{cases} 1, & 1 \leq i \leq a, j = \tilde{i}; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\{e_{m+1}, \dots, e_{m+c}; e_{m+n-d+1}, \dots, e_{m+n}\}, \quad 0 \leq d \leq n, \quad 0 \leq c \leq \lfloor \frac{n-d}{2} \rfloor$$

is an \mathbb{F} -basis of $V_{\bar{1}}$ satisfying

$$\beta(e_i, e_j) = \beta(e_j, e_i) = \begin{cases} -1, & m+n-d+1 \leq i=j \leq m+n; \\ 0, & \text{otherwise.} \end{cases}$$

The 4-tuple (a, b, c, d) is called the β -dimension of V , denoted by $\beta\text{-dim } V = (a, b, c, d)$. V is *nondegenerate* (with respect to β) if $a = b$ and $c = 0$. V is *isotropic* if $a = 0$ and $d = 0$. Clearly, for any \mathbb{Z}_2 -graded subspace of L_{-1} , there exists a β -basis of it, which can extend to a generalized orthosymplectic basis of L_{-1} .

Now, suppose $L = W, S, H$ or K . Put

$$\mathfrak{V}^L = \{V \mid V \text{ is a nontrivial subspace of } L_{-1}\}.$$

$V \in \mathfrak{V}^L$ is called a *standard element* if V is spanned by

$$\{\partial_1, \dots, \partial_k \mid \partial_{m+1}, \dots, \partial_{m+l}\},$$

when $L = W$ or S , $0 \leq k \leq m$, $0 \leq l \leq n$; if V is spanned by

$$\{y_1, \dots, y_a, y_{r+1}, \dots, y_{r+a}; y_{a+1}, \dots, y_b \mid y_{m+1}, \dots, y_{m+c}; y_{m+n-d+1}, \dots, y_{m+n}\},$$

when $L = H$ or K , $0 \leq a \leq b \leq r$, $0 \leq d \leq n$, $0 \leq c \leq \lfloor \frac{n-d}{2} \rfloor$. Hereafter, for $V, V' \in \mathfrak{V}^L$, the symbol $V \cong V'$ always means $\text{superdim } V = \text{superdim } V'$ when $L = W$ or S and means $\beta\text{-dim } V = \beta\text{-dim } V'$ when $L = H$ or K .

Lemma 2.2. *Let $L = W, S, H$ or K . Suppose $V, V' \in \mathfrak{V}^L$ satisfying $V \cong V'$. Then there exists a \mathbb{Z} -homogeneous automorphism Φ_L of L such that $\Phi_L(V) = V'$.*

Proof. Without loss of generality, we may assume that V is a standard element in \mathfrak{V}^L . When $L = W$ or S , suppose $\text{superdim } V = \text{superdim } V' = (k, l)$. Let $(E_1, \dots, E_k \mid E_{m+1}, \dots, E_{m+l})$ be a \mathbb{Z}_2 -homogeneous basis of V' . It extends to a \mathbb{Z}_2 -homogeneous basis of W_{-1} :

$$(E_1, \dots, E_m \mid E_{m+1}, \dots, E_{m+n}),$$

where $|E_i| = |\partial_i|$, $i \in \mathbf{I}$. There exists an even invertible matrix A_{m+n} such that

$$(E_1, \dots, E_{m+n}) = (\partial_1, \dots, \partial_{m+n})A^t. \quad (2.5)$$

Let $(\xi_1, \dots, \xi_{m+n}) = (x_1, \dots, x_{m+n})A^{-1}$. Consider the mapping ϕ such that

$$\phi(x_i) = \xi_i \quad \text{for all } i \in \mathbf{I}.$$

Notice that $|x_i| = |\xi_i|$, since A is even. By [20, Lemma 2.5], ϕ can extend to an endomorphism of \mathcal{O} , which is still written as ϕ . Then we have:

$$(\phi^{-1}(x_1), \dots, \phi^{-1}(x_{m+n})) = (x_1, \dots, x_{m+n})A \quad (2.6)$$

We denote by Φ the automorphism of W which is induced by ϕ according to the formula

$$\Phi(D) = \phi D \phi^{-1} \quad \text{for } D \in W.$$

Clearly, Φ is \mathbb{Z} -homogeneous. By (2.5) and (2.6), we have:

$$\Phi(\partial_i) = \phi \partial_i \phi^{-1} = E_i \quad \text{for all } i \in \mathbf{I}. \quad (2.7)$$

Furthemore, for $D = \sum_{i \in \mathbf{I}} f_i \partial_i \in W$, one can verify that

$$\Phi(D) = \phi D \phi^{-1} = \sum_{i,j \in \mathbf{I}} \partial_i(\phi^{-1}(x_j)) \phi(f_i) \partial_j.$$

By virtue of (2.5) and (2.7), we have:

$$\text{div}(\Phi(D)) = \phi(\text{div} D).$$

This shows that $\Phi(S) = S$ since $S = [\bar{S}, \bar{S}]$. Then $\Phi_L = \Phi|_L$ is desired.

When $L = H$ or K , suppose $\beta\text{-dim } V = \beta\text{-dim } V' = (a, b, c, d)$. Let $\{e_i \mid i \in \overline{1, 2r} \cup \overline{m+1, m+n}\}$ be an extension of β -basis (2.4) of V' to a generalized orthosymplectic basis of L_{-1} . Then, there exist two even invertible matrices

$$A = \left(\begin{array}{c|c} A_{2r} & 0 \\ \hline 0 & A_n \end{array} \right) \quad \text{and} \quad A' = \left(\begin{array}{c|c|c} A_{2r} & 0 & \\ \hline 0 & I_1 & 0 \\ \hline 0 & & A_n \end{array} \right)$$

satisfying

$$\begin{aligned} (e_1, \dots, e_{2r} \mid e_{m+1}, \dots, e_{m+n})A &= (y_1, \dots, y_{2r} \mid y_{m+1}, \dots, y_{m+n}), \\ (e_1, \dots, e_{2r}, e_{2r+1} \mid e_{m+1}, \dots, e_{m+n})A' &= (y_1, \dots, y_{2r}, y_{2r+1} \mid y_{m+1}, \dots, y_{m+n}). \end{aligned}$$

Thus, we obtain that

$$A^{-1}J(A^t)^{-1} = J. \quad (2.8)$$

By virtue of [20, Lemma 2.5], there exists a unique automorphism of \mathcal{O} denoted by ϕ_L satisfying $\phi_L(y_i) = e_i$, $i \in \mathbf{I}$. As in the case $L = W$, we denote by $\bar{\Phi}_L$ the \mathbb{Z} -homogeneous automorphism of W which is induced by ϕ_L . From (2.8), we have:

$$(\bar{\Phi}_H(D_1), \dots, \bar{\Phi}_H(D_{m+n})) = (D_1, \dots, D_{m+n})A^t. \quad (2.9)$$

$$(\bar{\Phi}_K(D_1), \dots, \bar{\Phi}_K(D_{m+n})) = (D_1, \dots, D_{m+n})A'^t. \quad (2.10)$$

For any $D = \sum_{i \in \mathbf{I} \setminus \{2r+1\}} f_i D_i \in W$ and $f D_{2r+1} \in W$, from (2.8)-(2.10) we have:

$$\bar{\Phi}_L(D) = \sum_{i \in \mathbf{I} \setminus \{2r+1\}} \phi_L(f_i) \bar{\Phi}_L(D_i), \quad (2.11)$$

$$\bar{\Phi}_K(f D_{2r+1}) = \phi_K(f) D_{2r+1}. \quad (2.12)$$

For any $f \in \mathcal{O}$, we have:

$$\bar{\Phi}_K(D_{2r+1}(f)\mathfrak{D}) = D_{2r+1}(\phi_K(f))\mathfrak{D}. \quad (2.13)$$

$$\bar{\Phi}_K((2 - \mathfrak{D})(f)D_{2r+1}) = (2 - \mathfrak{D})\phi_K(f)D_{2r+1}. \quad (2.14)$$

By virtue of (2.8)-(2.14), we have:

$$\bar{\Phi}_L(E_L(f)) = E_L(\phi_L(f)) \quad \text{for any } f \in \mathcal{O}.$$

It follows that $\bar{\Phi}_L(L) = L$ since $L = [\bar{L}, \bar{L}]$. Then $\Phi_L = \bar{\Phi}_L|_L$ is desired. \square

For convenience, we introduce the following notations. Let $L = W$ or S . For any $V \in \mathfrak{V}^L$ with $\text{superdim} V = (k, l)$, put

$$\mathbf{I}(k, l) = \overline{1, k} \cup \overline{m+1, m+l}, \quad \bar{\mathbf{I}}(k, l) = \mathbf{I} \setminus \mathbf{I}(k, l),$$

If V is standard, we have:

$$V = \text{span}_{\mathbb{F}}\{\partial_i \mid i \in \mathbf{I}(k, l)\}. \quad (2.15)$$

Let $L = H$ or K . For any $V \in \mathfrak{V}^L$ with $\beta\text{-dim } V = (a, b, c, d)$, put

$$\begin{aligned} I_{01} &= \overline{1, a}; \quad \bar{I}_{01} = \overline{r+1, r+a}; \quad I_{02} = \overline{a+1, b}; \quad \bar{I}_{02} = \overline{r+a+1, r+b}; \\ I_{11} &= \overline{m+n-d+1, m+n}; \quad I_{12} = \overline{m+1, m+c}; \quad \bar{I}_{12} = \overline{m+q+1, m+q+c}; \\ I_{03} &= \overline{b+1, r} \cup \overline{r+b+1, m}; \quad I_{13} = \overline{m+c+1, m+q} \cup \overline{m+q+c+1, m+n-d}. \end{aligned} \quad (2.16)$$

Obviously, $\mathbf{I} = J_1 \cup J_2 \cup \bar{J}_2 \cup J_3$, where

$$J_1 = I_{01} \cup \bar{I}_{01} \cup I_{11}, \quad J_2 = I_{02} \cup I_{12}, \quad \bar{J}_2 = \bar{I}_{02} \cup \bar{I}_{12} \quad \text{and} \quad J_3 = I_{03} \cup I_{13}. \quad (2.17)$$

We call J_i to be *single* (resp. *twinned*) if $\mathbf{I}_0 \cap J_i = \emptyset$ and there exists only one element in $\mathbf{I}_1 \cap J_i$ (resp. there exist two elements in $\mathbf{I}_1 \cap J_i$), $i = 1, 3$. If V is standard, we have:

$$V = \text{span}_{\mathbb{F}}\{y_i \mid i \in J_1 \cup J_2\}. \quad (2.18)$$

For any $i \in \mathbf{I}$, let us assign to each y_i a value as follows:

$$\nu(y_i) = \begin{cases} 1 & i \in J_1; \\ 0 & i \in J_2; \\ \frac{1}{3} & i \in \bar{J}_2; \\ 2 & i \in J_3. \end{cases} \quad (2.19)$$

If $a = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_m^{\alpha_m} y^u$, define $\nu(a) = \prod_{i \in \mathbf{I}_0} \nu(y_i)^{\alpha_i} \prod_{i \in u} \nu(y_i)$.

Remark 2.3. Let T be a torus of L , $L = W, S, H$ or K . Consider the weight space decompositions with respect to T :

$$L = L^\theta \oplus \bigoplus_{\gamma \in \Delta} L^\gamma, \quad L_i = L_i^\theta \oplus \bigoplus_{\gamma \in \Delta_i} L_i^\gamma,$$

where $\Delta_i \subset \Delta \subset T^*$ and θ is the zero weight. Notice the standard facts below.

- (1) For $t \in T$, suppose $x = x_1 + x_2 + \cdots + x_n \in L$ is a sum of eigenvectors of $\text{ad } t$ associated with mutually distinct eigenvalues. Then all x_i 's lie in $\text{alg}(\{t, x\})$.
- (2) $T = \text{span}_{\mathbb{F}}\{y_i y_{\bar{i}} \mid i \in \overline{1, r} \cup \overline{m+1, m+q}\}$ is a torus of L , $L = H$ or K , where $0 \leq d \leq n$, $q = \lfloor \frac{n-d}{2} \rfloor$. Define ϵ_j to be the linear function on T by

$$\epsilon_j(y_i y_{\bar{i}}) = \delta_{ji} - \delta_{\bar{j}\bar{i}}.$$

For $i, j, k \in \mathbf{I} \setminus \{2r+1\} \cup \overline{m+2q+1, m+n}$, if $\epsilon_i + \epsilon_j \in \Delta_0$, we have:

$$\dim L_{-1}^{\epsilon_i} = 1; \quad \dim L_0^{\epsilon_i + \epsilon_j} = 1; \quad L_0^{\epsilon_k} = \sum_{l=m+2q+1}^{m+n} \mathbb{F} y_k y_l.$$

3. MGS of Type (I)

To formulate the MGS of type (I), we introduce the following notations.

For $i \geq 1$, write

$$L'_i = \overline{S}_i = \{D \in L_i \mid \operatorname{div} D = 0\} \quad \text{and} \quad L''_i = \{f\mathfrak{D} \mid f \in \mathcal{O}_i\}, \quad (3.20)$$

where $L = W$ or S , \mathfrak{D} is the *degree derivation* of \mathcal{O} ; that is, $\mathfrak{D} = \sum_{k \in \mathbf{I}} x_k \partial_k$. Clearly, both W'_i and W''_i are nontrivial subspaces of W_i .

For $i \geq 0$, write

$$K_{ij} = \{u \in K_i \mid u = fz^j, f \in \mathcal{O}_{i+2-2j}, [1, f] = 0\}.$$

Clearly, K_{ij} is a nontrivial subspace of K_i .

Theorem 3.1. *All MGS of type (I) are characterized as follows:*

- (1) *If $m - n + 1 \equiv 0 \pmod{p}$ then W has exactly one MGS of type (I) :*

$$W_{-1} + W_0 + W'_1 + W'_2 + \cdots + W'_{\xi-2}$$

with dimension $(m + n - 1)2^n p^m + 2$;

If $m - n + 1 \not\equiv 0 \pmod{p}$ then W has exactly two MGS of type (I) :

$$W_{-1} + W_0 + W''_1 \quad \text{and} \quad W_{-1} + W_0 + W'_1 + W'_2 + \cdots + W'_{\xi-2}$$

with dimensions $(m + n)(m + n + 2)$ and $(m + n - 1)2^n p^m + 2$, respectively.

- (2) *If $m - n + 1 \equiv 0 \pmod{p}$ then S has exactly one MGS of type (I) :*

$$S_{-1} + S_0 + S''_1$$

with dimension $(m + n)^2 + 2(m + n) - 1$;

If $m - n + 1 \not\equiv 0 \pmod{p}$ then S has exactly one MGS of type (I) :

$$S_{-1} + S_0$$

with dimension $(m + n)^2 + (m + n) - 1$.

- (3) *H has exactly one MGS of type (I) :*

$$H_{-1} + H_0$$

with dimension $(m + n)^2 + m$.

- (4) *K has exactly two MGS of type (I) :*

$$K_{-2} + K_{-1} + K_0 + \sum_{i=1}^{2r(p-1)+n} K_{i0} \quad \text{and} \quad K_{-2} + K_{-1} + K_0 + K_{11} + K_{22}$$

with dimensions $2^n p^{2r} + 1$ and $(2r + n)^2 + 4r + n + 3$, respectively.

We note that many preliminary results in this section are analogous to the ones of Lie algebras (see [16, 17, 18]). We will need the following formulas which are easy to verify by direct calculations.

Lemma 3.2. For $f \in \mathcal{O}_s$ and $g \in \mathcal{O}_t$,

$$\begin{aligned} \operatorname{div}(f\mathfrak{D}) &= (m - n + s)f \quad \text{for } f \in \mathcal{O}_s, \\ [f\mathfrak{D}, g\mathfrak{D}] &= (t - s)fg\mathfrak{D}. \end{aligned} \quad (3.21)$$

Lemma 3.3. The following statements hold.

- (1) W'_s and W''_s are W_0 -submodules of W_s . Moreover, W''_s is irreducible.
- (2) If $m - n + s \not\equiv 0 \pmod{p}$ then $W_s = W'_s \oplus W''_s$;
- (3) If $m - n + s \equiv 0 \pmod{p}$ then $W''_s \subset W'_s$.

Proof. Note that div is a derivation from W to \mathcal{O} as W -module. Thus, (1), (2) and (3) hold by virtue of Lemma 3.2. \square

Below, the 1-component W_1 will be a focus of our attention. For convenience, we introduce two concepts, by which our arguments are largely simplified: An element \mathcal{L} in W_1 is called a *leader* if it is of the form

$$\mathcal{L} = x_1^2 \partial_1 + \sum_{i=2}^{m+n} f_i \partial_i \quad \text{where } f_i \in \mathcal{O}_2;$$

An element in W_1 is called *1-defective* if it is of the form

$$\sum_{i=2}^{m+n} f_i \partial_i \quad \text{where } f_i \in \mathcal{O}_2.$$

Lemma 3.4. Let $D \in W_1$.

- (1) $[x_1 \partial_j, D] = 0$ for all $j \geq 2$ if and only if $D = \lambda x_1 \mathfrak{D} + x_1^2 \sum_{j \geq 2} k_j \partial_j$ for some $\lambda, k_j \in \mathbb{F}$.
- (2) $[x_1 \partial_j, D] \in W''_1$ for all $j \geq 2$ if and only if $D = f \mathfrak{D} + x_1^2 \sum_{j \geq 2} k_j \partial_j$ for some $f \in \mathcal{O}_1$ and $k_j \in \mathbb{F}$.

Proof. (1) Suppose $[x_1 \partial_j, D] = 0$ for all $j \geq 2$ and write $D = \sum_i a_i \partial_i$. Then

$$x_1 \partial_j(a_i) - \delta_{i=j} (-1)^{|x_1 \partial_j| |a_1 \partial_1|} a_1 = 0 \quad \text{for all } j \geq 2. \quad (3.22)$$

Then $a_1 = kx_1^2$ for some $k \in \mathbb{F}$. If $k = 0$, it follows from (3.22) that $\partial_j(a_i) = 0$ for all $j, i \geq 2$. That is, $a_j = k_j x_1^2$ for all $j \geq 2$. Hence $D = x_1^2 \sum_{j \geq 2} k_j \partial_j$. If $k \neq 0$ then write $a_1 = x_1^2$. From (3.22) one deduces

$$\partial_j(a_i) = \delta_{i=j} x_1 \quad \text{for all } i, j \geq 2.$$

It follows that $a_j = k_j x_1^2 + x_1 x_j$ for $j \geq 2$ and one direction holds. The other one is clear.

(2) Write $[x_1 \partial_j, D] = f_j \mathfrak{D}$, $f_j \in \mathcal{O}_1$, $j \geq 2$. By acting on x_i with $i \neq 1, j$, we have $x_1 [\partial_j, D](x_i) = f_j x_i$ and then $f_j = k_j x_1$ for some $k_j \in \mathbb{F}$. Thus

$$[x_1 \partial_j, D] = k_j x_1 \mathfrak{D} \quad \text{for all } j \geq 2. \quad (3.23)$$

Since $[x_1 \partial_j, x_i \mathfrak{D}] = \delta_{i=j} x_1 \mathfrak{D}$, from (3.23) we have $[x_1 \partial_j, D - (\sum_{i \geq 2} k_i x_i) \mathfrak{D}] = 0$ for all $j \geq 2$. Now the conclusion follows from (1). \square

Lemma 3.5. *Let M be a nonzero W_0 -submodule of W_1 .*

- (1) *M contains a leader.*
- (2) *If M contains a leader which does not lie in W_1'' then M contains a nonzero 1-defective element.*
- (3) *If M contains a nonzero 1-defective element then $M \supset W_1'$. In particular, as W_0 -module, W_1' is generated by $x_1^2 \partial_j$ for any fixed $j \geq 2$.*
- (4) *As W_0 -module, W_1 is generated by $x_1^2 \partial_1$.*
- (5) *Any nonzero W_0 -submodule of W_1 different from W_1'' must contain W_1' .*

Proof. (1), (3) and (4) need only a straightforward verification.

(2) Let $D = x_1^2 \partial_1 + \dots$ be a leader in $M \setminus W_1''$. Then $[x_1 \partial_j, D] \in M$ are 1-defective for all $j \geq 2$. If they are not all zero, we are done. Otherwise, by Lemma 3.4(1),

$$D = x_1 \mathfrak{D} + x_1^2 \sum_{j \geq 2} k_j \partial_j \quad \text{for some } k_j \in \mathbb{F}.$$

Clearly, $\sum_{j \geq 2} k_j \partial_j \neq 0$, say, $k_2 \neq 0$. Consequently, $x_1^2 \partial_2 = k_2^{-1} [x_2 \partial_2, D] \in M$.

(5) Let M be a nonzero W_0 -submodule and $M \neq W_1'$. Let us show that $M \supset W_1'$. By (1), (2) and (3) we may assume that all the leaders of M lie in W'' . Then $W'' \subset M$, since W'' as W_0 -module is irreducible by Lemma 3.3(1). For $D \in M \setminus W''$, if there is some $i \geq 2$ such that $E = [x_1 \partial_i, D] \notin W''$, then $[x_1 \partial_j, E]$ is a leader or 1-defective for any $j \geq 2$. By (3), one may assume that there is $D \in M \setminus W''$ which is pulled into W'' by any $x_1 \partial_j$ with $j \geq 2$. Then by Lemma 3.4(2), M contains a nonzero 1-defective element and then $M \supset W'$. \square

Lemma 3.6. *The following statements hold.*

- (1) *W_1' is a maximal W_0 -submodule of W_1 .*
- (2) *If $m - n + 1 \not\equiv 0 \pmod{p}$, W_0 -module W_1' is irreducible. In particular, W_1 has a decomposition of irreducible W_0 -submodules:*

$$W_1 = W_1' \oplus W_1''.$$

- (3) *If $m - n + 1 \equiv 0 \pmod{p}$, W_1 has exactly a composition series of W_0 -submodules:*

$$0 \subset W_1'' \subset W_1' \subset W_1.$$

Proof. (1) Let M be a submodule of W_1 containing strictly W_1' . Note that

$$\text{div} : \text{span}_{\mathbb{F}}\{x_1 x_1 \partial_1, x_2 x_1 \partial_1, \dots, x_{m+n} x_1 \partial_1\} \mapsto \mathcal{O}_1$$

is surjective. Pick any $D \in M \setminus W_1'$. Then there exists $E = f x_1 \partial_1$, $f \in \mathcal{O}_1$, such that $\text{div} E = \text{div} D$. That is, $E - D \in W_1' \subset M$ and then $0 \neq E \in M$. If $\partial_j(f) = 0$ for all $j \geq 2$ then $E = \partial_1(f) x_1^2 \partial_1$ and hence $M = W_1$ by Lemma 3.5(4). Suppose $\partial_j(f) \neq 0$ for some $j \neq 1$. Then $x_j x_1 \partial_1 = \partial_j(f)^{-1} [x_j \partial_j, E] \in M$. It follows that

$$x_1^2 \partial_1 - (-1)^{|\partial_j|} x_j x_1 \partial_j = [x_1 \partial_j, x_j x_1 \partial_1] \in M.$$

Note that $\frac{1}{2} x_1^2 \partial_1 - (-1)^{|\partial_j|} x_j x_1 \partial_j$ is in $W_1' \subset W$. It follows that $x_1^2 \partial_1 \in M$ and $M = W_1$ by Lemma 3.5(4), showing that W_1' is maximal.

- (2) and (3) are immediate consequences of Lemmas 3.3 and 3.5(5). \square

Corollary 3.7. *The following statements hold.*

- (1) *If $m - n + 1 \not\equiv 0 \pmod{p}$ then S_1 is an irreducible S_0 -module.*
- (2) *If $m - n + 1 \equiv 0 \pmod{p}$ then S_1'' is the unique nontrivial S_0 -submodule of S_1 .*

Proof. Note that $W_0 = S_0 + \mathbb{F}\mathfrak{D}$ and that $W_1' = S_1$. If $m - n + 1 \equiv 0 \pmod{p}$ then $S_1'' = W_1''$. The lemma follows directly from Lemma 3.6. \square

Lemma 3.8. $\{D \in W_2 \mid [W_{-1}, D] \subset W_1''\} = 0$.

Proof. Write $D = \sum_{i \in \mathbf{I}} a_i \partial_i \in W_2$ and suppose D is pulled into W_1'' by W_{-1} . Then, each a_i must be a multiple of x_i^2 and in particular, $a_j = 0$ for all $j > m$. Write $D = \sum_{i \leq m} f_i x_i^2 \partial_i$, where $f_i \in \mathcal{O}_1$. Since $[\partial_j, D] \in W_1''$, one deduces that $\partial_j(f_i) = 0$ for $j > m$ and $i \leq m$. Now it is clear that D is not in W_1'' unless it is zero. \square

Let

$$\begin{aligned} M' &= W_{-1} + W_0 + W_1' + W_2' + \cdots + W_{\xi-2}', \\ M'' &= W_{-1} + W_0 + W_1''. \end{aligned}$$

Using (3.21) and keeping in mind that div is a derivation from W to \mathcal{O} , one may verify that M' and M'' are subalgebras of W .

Lemma 3.9. *Suppose M is a proper subalgebra containing $W_{-1} \oplus W_0 \oplus W_1'$. Then $M \subset M'$.*

Proof. Assume conversely that $M \not\subset M'$. Then there exists $D \in M \cap \sum_{i \geq 1} W_i$ satisfying $\text{div} D \neq 0$. Using the formula $\text{div}[\partial_j, D] = \partial_j(\text{div} D)$ for all $j \in \mathbf{I}$, one sees that $M \supset W_1$ by Lemma 3.6(1). By Lemma 2.1(2), $M = W$, a contradiction. \square

Proof of (1) and (2) in Theorem 3.1 (1) *Claim A:* M' is maximal. This follows immediately from Lemma 3.9.

Claim B: M'' is maximal if $m - n + 1 \not\equiv 0 \pmod{p}$. Let M be a subalgebra strictly containing M'' . By transitivity and Lemma 3.8, $M \cap W_1$ must strictly contain W_1'' . Lemma 3.6(2) forces $M \supset W_1$ and therefore, $M = W$ by Lemma 2.1(2).

Claim C: M' and M'' exhaust all the maximal subalgebras of type (I). Let M be a maximal subalgebra of type (I). By transitivity, M must contain a nonzero element of W_1 and therefore, $M \cap W_1 \neq 0$ is a nonzero W_0 -submodule of W_1 . By Lemma 3.5(5), we have $M \cap W_1 = W_1''$ or $M \cap W_1 \supset W_1'$.

Case 1. Suppose $m - n + 1 \not\equiv 0 \pmod{p}$. If $M \cap W_1 = W_1''$ then Claim B forces $M = M''$. Suppose $M \cap W_1 \supset W_1'$. By Lemma 3.9, we have $M \subset M'$ and then $M = M'$ by the maximality of M .

Case 2. Suppose $m - n + 1 \equiv 0 \pmod{p}$. We have $M \supset W''$. Since $W'' \subsetneq W'$ in this situation, one sees $M \supsetneq W''$. By transitivity and Lemma 3.8, $M \cap W_1 \supsetneq W_1''$ and hence $M \cap W_1 \supset W_1'$ by Lemma 3.5(5). It follows from Lemma 3.9 that $M = M'$. This completes the proof of (1).

(2) First of all, $S_{-1} + S_0$ and $S_{-1} + S_0 + S_1''$ ($m + n - 1 \equiv 0 \pmod{p}$) are subalgebras of S . Let M be a maximal subalgebra of S containing $S_{-1} + S_0$. Note

that $W_1'' = S_1''$ when $m - n + 1 \equiv 0 \pmod{p}$. By the transitivity of S , Lemmas 2.1(2), 3.8 and Corollary 3.7, we obtain that $M = S_{-1} + S_0 + S_1''$ when $m + n - 1 \equiv 0 \pmod{p}$; $M = S_{-1} + S_0$ when $m + n - 1 \not\equiv 0 \pmod{p}$. The process shows also that these two subalgebras are indeed maximal. This completes the proof of (2). \square

Remark 3.10. For W and S , the arguments for MGS of the other types will be reduced to the case of type (I) MGS by the method of minimal counterexample.

Lemma 3.11. The following statements hold.

- (1) H_1 is an irreducible H_0 -module.
- (2) For $i \geq 0$, K_i is a direct sum of K_0 -submodules K_{ij} . Moreover, K_{10} and K_{11} are irreducible K_0 -modules.

Proof. Using the results in the case of modular Lie algebras [17] and by a direct computation, it is easy to show that (1) holds. Since $K_{10} \cong H_1$ and $K_{11} \cong H_{-1}$ as H_0 -modules, by irreducibilities of H_{-1} and H_1 , (2) holds. \square

Let

$$\begin{aligned} M' &= K_{-2} + K_{-1} + K_0 + \sum_{i=1}^{2r(p-1)+n} K_{i0}, \\ M'' &= K_{-2} + K_{-1} + K_0 + K_{11} + K_{22}. \end{aligned}$$

By a standard and direct computation, one may verify that M' and M'' are subalgebras of K .

Proof of (3) and (4) in Theorem 3.1 (3) This statement follows immediately from Lemmas 2.1(2) and 3.11(1).

(4) *Claim A:* M' is maximal. For any $0 \neq u \in K$, $u \notin M'$, put $\overline{M} = \text{alg}(M' + \mathbb{F}u)$. Note that there exist $k \in \mathbb{N}$ and $v_1, \dots, v_s \in K_{-1}$ such that

$$0 \neq u_1 = fz + \alpha z^2 = [v_1, [\dots [v_s, (\text{ad } 1)^k u] \dots]] \in \overline{M} \setminus M',$$

where $f \in \mathcal{O}$ satisfying $[1, f] = 0$ and $\alpha \in \mathbb{F}$. Then there exists $i \in \mathbf{I}$ such that $[y_i, u_1] - y_i f \neq 0$. It follows that

$$0 \neq (\sigma(\tilde{i})(-1)^i D_i(f) + \alpha y_i)z \in \overline{M}.$$

From Lemma 2.1(1), there exists a nonzero element in $\overline{M} \cap K_{11}$. By Lemmas 2.1(2) and 3.11(2), we have $\overline{M} = K$. Thus M' is maximal.

Claim B: M'' is maximal. For any $0 \neq u \in K$, $u \notin M''$, put $\overline{M} = \text{alg}(M'' + \mathbb{F}u)$. It is sufficient to show that there exists a nonzero element in $K_{10} \cap \overline{M}$. When $\text{zd}(u) > 2$, by transitivity, there exist $v_1, \dots, v_s \in K_{-1}$ such that

$$0 \neq u_3 = u_{30} + u_{31} + u_{32} = [v_1, [\dots [v_s, u] \dots]] \in \overline{M},$$

where $u_{3i} \in K_{3i}$, $i = 0, 1, 2$. Note that $[1, u_{32}] \in K_{11}$. If $u_{31} \neq 0$, then

$$0 \neq [1, u_{31}] = [1, u_3 - u_{30} - u_{32}] \in \overline{M} \cap K_{10}.$$

If $u_{31} = 0$, there exists $j \in \mathbf{I}$ such that

$$0 \neq u_2 = \sigma(\tilde{j})(-1)^j D_j(u_{30}) + y_{\tilde{j}} D_{2r+1}(u_{32}) \in \overline{M} \setminus M'',$$

Note that $\text{zd}(u_2) = 2$. Thus, it remains to consider the case $\text{zd}(u) = 2$. Assume that $u = u_{20} + u_{21}$, where $u_{2i} \in K_{2i}$, $i = 0, 1$. If $u_{21} = 0$ the conclusion follows. Notice that $H_0 \cong K_{21}$ as H_0 -module. If $u_{21} \neq 0$, by Remark 2.3 and a direct computation, we obtain that there exists $i \in \mathbf{I}_0$ such that $u_2 = u'_{20} + y^{(2\varepsilon_i)} z \in \overline{M}$, where $u'_{20} \in K_{20}$. Since $[K_{-1}, u_2] \subset \overline{M}$, there exists $j \in \mathbf{I}$ such that

$$0 \neq \sigma(\tilde{j})(-1)^j D_j(u'_{20}) + y_{\tilde{j}} y^{(2\varepsilon_i)} \in \overline{M} \cap K_{10}.$$

Thus the conclusion holds.

Claim C: M' and M'' exhaust all the maximal graded subalgebras of type (I). Suppose N is a maximal graded subalgebra of K containing $K_{-1} + K_0$. By transitivity, there exists $0 \neq D = D_{10} + D_{11} \in N$, where $D_{1i} \in K_{1i}$, $i = 0, 1$. Since $N \subsetneq K$, we claim that $D_{11} = 0$ or $D_{10} = 0$. Indeed, if $D_{11} \neq 0$ and $D_{10} \neq 0$, by the irreducibility of K_{10} , we have

$$w = y^{(2\varepsilon_1)} y_{\tilde{1}} + \sum_{i=1}^{2r+n} \alpha_i y_i z \in N, \quad \alpha_i \in \mathbb{F}.$$

We consider the following cases.

Case 1. For all i , $\alpha_i = 0$. Obviously, $N = K$ by the irreducibility of K_{1i} and $D_{1i} \neq 0$, $i = 0, 1$.

Case 2. There exists k such that $\alpha_k \neq 0$. If $k \neq 1, \tilde{1}$, for $\tilde{k} \neq j \in \mathbf{I}_1$, we have:

$$0 \neq [y_j y_{\tilde{k}}, w] \in N \cap K_{11}.$$

Similar to Case 1, we have $N = K$.

If $k = 1$ or $\tilde{1}$, then $w = y^{(2\varepsilon_1)} y_{\tilde{1}} + \alpha_1 y_1 z + \alpha_{\tilde{1}} y_{\tilde{1}} z$. For $j \in \mathbf{I}_1$, we have:

$$y_j y_1 y_{\tilde{1}} = [[y_j y_1, w], y^{(2\varepsilon_{\tilde{1}})}] \in N \cap K_{10}.$$

Similar to Case 1, we have $N = K$.

Consequently, $N = M'$ when $D_{11} = 0$ and $N = M''$ when $D_{10} = 0$. \square

4. MGS of Type (II)

Let $L = W, S, H$ or K . Recall

$$\mathfrak{V}^L = \{V \mid V \text{ is a nontrivial subspace of } L_{-1}\}.$$

To describe the MGS of type (II) of L , for any $V \in \mathfrak{V}^L$, we define

$$\mathcal{M}(V) = \oplus_{i \geq -2} \mathcal{M}_i(V),$$

where

$$\begin{aligned} \mathcal{M}_{-1}(V) &= V; \quad \mathcal{M}_{-2}(V) = [\mathcal{M}_{-1}(V), \mathcal{M}_{-1}(V)]; \\ \mathcal{M}_i(V) &= \{u \in L_i \mid [V, u] \subset \mathcal{M}_{i-1}(V)\} \quad \text{for } i \geq 0. \end{aligned} \quad (4.24)$$

Theorem 4.1. *Suppose $L = W$ or S . All MGS of type (II) of L are characterized as follows:*

- (1) *All MGS of type (II) of L are precisely:*

$$\{\mathcal{M}(V) \mid V \in \mathfrak{V}^L\}.$$

- (2) *For any V and V' in \mathfrak{V}^L ,*

$$\mathcal{M}(V) \stackrel{\text{algebra}}{\cong} \mathcal{M}(V') \iff V \cong V'.$$

- (3) *L has exactly $(m+1)(n+1) - 2$ isomorphism classes of MGS of type (II).*

- (4) *If $\text{superdim} V = (k, l)$, then*

$$\dim \mathcal{M}(V) = \begin{cases} 2^n p^m (m+n) - 2^l p^k (m+n-k-l), & L = W; \\ 2^n p^m (m+n-1) + 1 - 2^l p^k (m+n-k-l), & L = S. \end{cases}$$

When $L = H$ or K , recall definitions (2.16)-(2.18) mentioned in Section 2. Put

$$\mathcal{V}^L = \{V \in \mathfrak{V}^L, \text{ satisfying } J_3 \text{ is neither single nor twinned}\};$$

$$\mathcal{W}^K = \{V \in \mathfrak{V}^K, \text{ satisfying } J_3 \text{ is not twinned}\}.$$

Suppose $V \in \mathfrak{V}^K$ is isotropic. Put

$$\mathcal{M}^K(1, V) = \oplus_{i \geq -2} \mathcal{M}_i^K(1, V),$$

where

$$\mathcal{M}_{-2}^K(1, V) = \mathbb{F}; \quad \mathcal{M}_{-1}^K(1, V) = V;$$

$$\mathcal{M}_i^K(1, V) = \{u \in K_i \mid [V, u] \subset \mathcal{M}_{i-1}^K(1, V), [1, u] \in \mathcal{M}_{i-2}^K(1, V)\}, \quad i \geq 0.$$

Theorem 4.2. *All MGS of type (II) of H and K are characterized as follows:
For H ,*

- (1) *All MGS of type (II) are precisely:*

$$\{\mathcal{M}(V) \mid V \in \mathcal{V}^H\}.$$

- (2) *For any V and V' in \mathcal{V}^H ,*

$$\mathcal{M}(V) \stackrel{\text{algebra}}{\cong} \mathcal{M}(V') \iff V \cong V'.$$

- (3) *H has exactly $\phi(r, n)$ isomorphism classes of MGS of type (II), where*

$$\phi(r, n) = \begin{cases} 8^{-1}(r+1)(r(n+2)^2 + 2n^2 + 6 - r) - 2, & n \text{ is odd}; \\ 8^{-1}(r+1)(r(n+2)^2 + 2n^2 + 8) - 2, & n \text{ is even}. \end{cases}$$

- (4) *For $V \in \mathcal{V}^H$, if $\beta\text{-dim} V = (a, b, c, d)$, then*

$$\dim \mathcal{M}(V) = p^m 2^n + p^{2a} 2^d - p^{a+b} 2^{c+d} (m - 2b + n - d - 2c + 1) - 2.$$

For K ,

(1') All MGS of type (II) are precisely:

$$\begin{aligned} & \{ \mathcal{M}(V) \mid V \in \mathcal{V}^K \text{ is neither nondegenerate nor isotropic} \} \\ & \cup \{ \mathcal{M}(V) \mid V \in \mathcal{W}^K \text{ is nondegenerate or isotropic} \} \\ & \cup \{ \mathcal{M}^K(1, V) \mid V \in \mathcal{V}^K \text{ is isotropic} \}. \end{aligned}$$

(2') For all MGS of type (II),

$$\begin{aligned} \mathcal{M}(V) & \stackrel{\text{algebra}}{\cong} \mathcal{M}(V') \iff V \cong V', \\ \mathcal{M}^K(1, V) & \stackrel{\text{algebra}}{\cong} \mathcal{M}^K(1, V') \iff V \cong V'. \end{aligned}$$

(3') K has exactly $\phi(r, n)$ isomorphism classes of MGS of type (II), where

$$\phi(r, n) = \begin{cases} 8^{-1}(r+1)(r(n+2)^2 + 2n^2 + 4n + 2 - r) + r - 1, & n \text{ is odd}; \\ 8^{-1}(r+1)(r(n+2)^2 + 2n^2 + 4n + 8) + r - 2, & n \text{ is even}. \end{cases}$$

(4') Let $\delta = 1$ when $n - m - 3 = 0 \pmod{p}$ and $\delta = 0$, otherwise. Suppose $\beta\text{-dim} V = (a, b, c, d)$.

(a) If V is isotropic, then

$$\begin{aligned} \dim \mathcal{M}(V) &= p^m 2^n - p^b 2^c (m - 2b + n - 2c) - \delta, \text{ when } V \in \mathcal{W}^K; \\ \dim \mathcal{M}^K(1, V) &= p^m 2^n - p^{b+1} 2^c (m - 2b + n - 2c) + p - \delta, \text{ when } V \in \mathcal{V}^K. \end{aligned}$$

(b) If V is not isotropic, then

$$\dim \mathcal{M}(V) = p^m 2^n - (2r - 2a + n - d)p^{2a+1} 2^d - p,$$

when $V \in \mathcal{W}^K$ is nondegenerate satisfying J_3 is single;

$$\dim \mathcal{M}(V) = p^m 2^n + p^{2a+1} 2^d - p^{a+b+1} 2^{c+d} (m - 2b + n - d - 2c) - \delta,$$

when $V \in \mathcal{W}^K$ is nondegenerate satisfying J_3 is not single or $V \in \mathcal{V}^K$ is not nondegenerate.

Lemma 4.3. Suppose $L = W, S, H$ or K .

(1) $\mathcal{M}(V)$ is a \mathbb{Z} -graded subalgebra of L . $\mathcal{M}^K(1, V)$ is a \mathbb{Z} -graded subalgebra of K .

(2) Suppose Φ is a \mathbb{Z} -homogeneous automorphism of L . Then

$$\Phi(\mathcal{M}_i(V)) = \mathcal{M}_i(\Phi(V)) \text{ for all } i \geq -2.$$

Moreover,

$$\Phi(\mathcal{M}(V)) = \mathcal{M}(\Phi(V)).$$

For K ,

$$\Phi(\mathcal{M}_i^K(1, V)) = \mathcal{M}_i^K(1, \Phi(V)) \text{ for all } i \geq -2.$$

Moreover,

$$\Phi(\mathcal{M}^K(1, V)) = \mathcal{M}^K(1, \Phi(V)).$$

- (3) If M is an MGS of type (II) of L , then $M = \mathcal{M}(M_{-1})$ unless $L = K$, M_{-1} is isotropic and $M_{-2} \neq 0$. However, in the latter case, $M = \mathcal{M}^K(1, M_{-1})$.

Proof. The approach is analogous to that used in the case of modular Lie algebras [16]. \square

Remark 4.4. Suppose $L = W, S, H$ or K . In view of Lemmas 2.2 and 4.3, for any $V \in \mathfrak{V}^L$, we may assume that V is standard [see (2.15), (2.18)].

Now, we consider the case $L = W$ or S . Suppose $V \in \mathfrak{V}^L$ with $\text{superdim} V = (k, l)$. For $L = W$, it is easy to verify that $\mathcal{M}_0(V)$ has a standard \mathbb{F} -basis $\mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\begin{aligned}\mathcal{A}_1 &= \{x_i \partial_j \mid i, j \in \mathbf{I}(k, l)\}, \\ \mathcal{A}_2 &= \{x_i \partial_j \mid i \in \bar{\mathbf{I}}(k, l), j \in \mathbf{I}\}.\end{aligned}$$

Similarly, for $L = S$, $\mathcal{M}_0(V)$ has a standard \mathbb{F} -basis $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, where

$$\begin{aligned}\mathcal{C}_1 &= \{x_i \partial_j \mid i, j \in \mathbf{I}(k, l), i \neq j\}, \\ \mathcal{C}_2 &= \{x_i \partial_j \mid i \in \bar{\mathbf{I}}(k, l), j \in \mathbf{I}, i \neq j\}, \\ \mathcal{C}_3 &= \{x_1 \partial_1 - (-1)^{|\partial_i|} x_i \partial_i \mid i \in \mathbf{I} \setminus \{1\}\}.\end{aligned}$$

Moreover, in any case of $L = W$ or S , $\mathcal{M}_0(V)$ has a standard co-basis in W_0 :

$$\mathcal{A}_3 = \{x_i \partial_j \mid i \in \mathbf{I}(k, l), j \in \bar{\mathbf{I}}(k, l)\}. \quad (4.25)$$

Lemma 4.5. Suppose $U, V \in \mathfrak{V}^L$, $L = W$ or S .

- (1) $\mathcal{M}_0(V)$ is a maximal subalgebra of L_0 .
- (2) $\mathcal{M}_0(U) = \mathcal{M}_0(V)$ if and only if $U = V$.

Proof. (1) Let \mathfrak{G}_0 be a subalgebra of L_0 which strictly contains $\mathcal{M}_0(V)$. It is clear that \mathfrak{G}_0 contains a nonzero element of form $B = \sum_{h,t \geq 1} \alpha_{ht} x_{i_h} \partial_{j_t}$, where $0 \neq \alpha_{ht} \in \mathbb{F}$, $i_h \in \mathbf{I}(k, l)$, $j_t \in \bar{\mathbf{I}}(k, l)$.

When $L = W$, for any $i \in \mathbf{I}(k, l)$ and $j \in \bar{\mathbf{I}}(k, l)$, one has $x_i \partial_{i_1} \in \mathcal{A}_1$ and $x_{j_1} \partial_j \in \mathcal{A}_2$. Then

$$x_i \partial_j = \alpha_{11}^{-1} [x_i \partial_{i_1}, [B, x_{j_1} \partial_j]] \in \mathfrak{G}_0,$$

showing that the co-basis $\mathcal{A}_3 \subset \mathfrak{G}_0$. Hence $\mathfrak{G}_0 = W_0$.

When $L = S$, suppose $|\mathbf{I}(k, l)| > 1$ and $|\bar{\mathbf{I}}(k, l)| > 1$. Choosing $x_{j_1} \partial_j$ in \mathcal{C}_2 with $j \in \bar{\mathbf{I}}(k, l) \setminus \{j_1\}$ and $x_i \partial_{i_1}$ in \mathcal{C}_1 with $i \in \mathbf{I}(k, l) \setminus \{i_1\}$, we have

$$x_i \partial_j = [x_i \partial_{i_1}, [B, x_{j_1} \partial_j]] \in \mathfrak{G}_0,$$

showing that the co-basis $\mathcal{A}_3 \subset \mathfrak{G}_0$ and then $\mathfrak{G}_0 = S_0$. For the remaining case $|\mathbf{I}(k, l)| = 1$ or $|\bar{\mathbf{I}}(k, l)| = 1$, the argument is similar and much easier.

(2) One direction is obvious. Note that one may choose bases of U and V as follows:

$$\overbrace{E_1, \dots, E_r}^{\text{cobasis in } U}, \overbrace{F_1, \dots, F_s}^{\text{basis of } U \cap V}, \overbrace{G_1, \dots, G_t}^{\text{cobasis in } V}$$

where $(E_1, \dots, E_r, F_1, \dots, F_s, G_1, \dots, G_t)$ is a permutation of ∂_i 's. Keeping in mind the standard co-basis (4.25), we are done by a similar argument as in (1). \square

Proposition 4.6. $\mathcal{M}(V)$ is maximal in L for any $V \in \mathfrak{V}^L$, $L = W$ or S .

Proof. Let M be an MGS containing $\mathcal{M}(V)$. Then $\mathcal{M}_i(V) \subset M_i$ for all $i \geq -1$. In particular, because of the maximality of $\mathcal{M}_0(V)$, it must be $M_0 = \mathcal{M}_0(V)$ or $M_0 = L_0$.

Case 1. Suppose $M_0 = \mathcal{M}_0(V)$. By induction, it is routine to verify that $M_i = \mathcal{M}_i(V)$ for all $i \geq 0$. Assume on the contrary that M strictly contains $\mathcal{M}(V)$. Then $M_{-1} \supsetneq \mathcal{M}_{-1}(V) = V$. Note that $\mathcal{M}_0(V) = M_0 = \mathcal{M}_0(M_{-1})$ from Lemma 4.3(3). Thus, Lemma 4.5(2) forces $M_{-1} = L_{-1}$. Pick any $i \in \mathbf{I}(k, l)$, $j \in \bar{\mathbf{I}}(k, l)$ and any $h \neq i, j$. We are able to check that

$$A = (-1)^{|x_h|} x_i x_j \partial_j - (-1)^{|x_j|} x_i x_h \partial_h \in S_1 \subset W_1.$$

Moreover, $A \in \mathcal{M}_1(V) = M_1$. Since $M_{-1} = W_{-1} = S_{-1}$, we have

$$x_i \partial_j = (-1)^{(|x_h| + |x_i||x_j|)} [\partial_j, A] \in M_0 = \mathcal{M}_0(V).$$

This contradicts the fact that $x_i \partial_j \in \mathcal{A}_3$ [see (4.25)]. Therefore, $M = \mathcal{M}(V)$.

Case 2. Suppose $M_0 = L_0$. In this case, since L_{-1} is irreducible as L_0 -module, we have $M_{-1} = L_{-1}$. Hence M is an MGS of type (I). By Theorem 3.1(1) and (2), $M_1 = W'_1, W''_1, S''_1$, or $\{0\}$. In Case 1, we have shown that $A \in \mathcal{M}_1(V)$. However, it is clear that A does not belong to W'_1, W''_1, S''_1 , or $\{0\}$. Hence $\mathcal{M}_1(V) \not\subset M_1$. This contradicts the assumption that M is a graded subalgebra containing $\mathcal{M}(V)$. \square

Proof of Theorem 4.1 (1), (2) and (3) are immediate consequences of Lemmas 2.2, 4.3(3) and Proposition 4.6. It remains to show the dimension formulas. For W , $\mathcal{M}(V)$ has a standard \mathbb{F} -basis which is a disjoint union:

$$\begin{aligned} & \{x^{(\alpha)} x^u \partial_i \mid \alpha \in \mathbf{A}(m), u \in \mathbf{B}(n); i \in \mathbf{I}(k, l)\} \\ & \cup \{x^{(\alpha)} x^u \partial_i \mid i \in \bar{\mathbf{I}}(k, l) \text{ and } \exists j \in \bar{\mathbf{I}}(k, l) \text{ such that } \partial_j(x^{(\alpha)} x^u) \neq 0\}. \end{aligned}$$

A standard and direct computation shows that:

$$\dim \mathcal{M}(V) = 2^n p^m (m + n) - 2^l p^k (m + n - k - l).$$

Similarly, for S , we have:

$$\dim \mathcal{M}(V) = 2^n p^m (m + n - 1) + 1 - 2^l p^k (m + n - k - l).$$

\square

Next, we consider the case $L = H$ or K . In this case, we shall frequently use the standard facts mentioned in Remark 2.3 without notice. Suppose $V \in \mathfrak{V}^L$ with $\beta\text{-dim } V = (a, b, c, d)$. In order to prove Theorem 4.2 we list the following assertions. For simplicity, we write $\lambda_{i,j}$ for a nonzero element in \mathbb{F} , where $i, j \in \mathbf{I}$. Recall definitions (2.16)-(2.19). Put

$$\begin{aligned} \mathcal{V}_i^L &= \{V \in \mathfrak{V}^L \mid V \text{ is isotropic and } J_3 \text{ is not twinned}\}; \\ \mathcal{V}_n^L &= \{V \in \mathfrak{V}^L \mid V \text{ is nondegenerate and } J_i \text{ is not twinned, } i = 1, 3\}; \\ \mathcal{V}_0^L &= \{V \in \mathfrak{V}^L \mid V \text{ is degenerate, } J_3 \text{ is empty and } J_1 \text{ is not twinned}\}. \end{aligned} \quad (4.26)$$

Lemma 4.7. For H , put

$$A_i = \text{span}_{\mathbb{F}}\{u \in H_i \mid \nu(u) = 0, 1 \text{ or } (\frac{1}{3})^k 2^l, \ k, l \in \mathbb{N}, \ l > 1\}.$$

Then

- (1) $A_i = \mathcal{M}_i(V)$, $i \geq -1$.
- (2) The subalgebra $A_0 = \mathcal{M}_0(V)$ is maximal in H_0 if and only if

$$V \in \mathcal{V}_i^H \cup \mathcal{V}_n^H \cup \mathcal{V}_0^H.$$

Proof. (1) It follows by using induction on i , $i \geq -1$.

(2) Obviously, the torus T mentioned in Remark 2.3(2) is contained in $\mathcal{M}_0(V)$. For any $h \in H_0$ and $h \notin \mathcal{M}_0(V)$, put $\overline{M} = \text{alg}(\mathcal{M}_0(V) + \mathbb{F}h)$. Firstly, we show the maximality of $\mathcal{M}_0(V)$. It suffices to prove $H_0 = \overline{M}$.

Case 1. $V \in \mathcal{V}_i^H$. Notice that

$$\nu(y_i) = \begin{cases} 0 & i \in J_2; \\ \frac{1}{3} & i \in \bar{J}_2; \\ 2 & i \in J_3 \end{cases} \quad \text{and} \quad \mathcal{M}_0(V) = \text{span}_{\mathbb{F}}\{y_i y_j \mid (i, j) \in J_2 \times \mathbf{I} \cup J_3 \times J_3\}.$$

We may assume that h is a monomial with $\nu(h) = \frac{1}{9}$ or $\frac{2}{3}$. When $h = y_i y_j$, $(i, j) \in \bar{J}_2 \times \bar{J}_2$, we have:

$$\begin{aligned} y_k y_l &= \lambda_{k,l} [y_i y_j, y_{\bar{j}} y_k], y_{\bar{i}} y_l \in \overline{M} \text{ for all } (k, l) \in \bar{J}_2 \times \bar{J}_2, \\ y_k y_s &= \lambda_{k,s} [y_k y_l, y_{\bar{l}} y_s] \in \overline{M} \text{ for all } s \in J_3. \end{aligned}$$

Thus, $H_0 = \overline{M}$. When $h = y_i y_j$, $(i, j) \in \bar{J}_2 \times J_3$, if $j \in I_{03}$ or I_{03} is not empty, we get $H_0 = \overline{M}$ in an analogous way as above. Otherwise, we may assume that I_{03} is empty. If J_3 is single, we have:

$$\begin{aligned} y_k y_{m+n-d} &= \lambda_{k,m+n-d} [y_i y_{m+n-d}, y_{\bar{i}} y_k] \in \overline{M} \text{ for all } k \in \bar{J}_2, \\ y_k y_l &= \lambda_{k,l} [y_k y_{m+n-d}, y_{m+n-d} y_l] \in \overline{M} \text{ for all } l \in \bar{J}_2. \end{aligned}$$

It follows that $H_0 = \overline{M}$. If J_3 is neither single nor twinned, for $s \in J_{13}$, $s \neq \tilde{j}$, we have:

$$\begin{aligned} y_k y_s &= \lambda_{k,s} [y_i y_j, y_{\bar{i}} y_k], y_{\bar{j}} y_s \in \overline{M} \text{ for all } k \in \bar{J}_2, \\ y_k y_{\tilde{j}} &= \lambda_{k,\tilde{j}} [y_k y_s, y_{\tilde{s}} y_{\tilde{j}}] \in \overline{M} \text{ } s \neq j \text{ and } \tilde{j}, \\ y_k y_l &= \lambda_{k,l} [y_j y_k, y_{\bar{j}} y_l] \in \overline{M} \text{ for all } l \in \bar{J}_2. \end{aligned}$$

Thus, $H_0 = \overline{M}$.

Case 2. $V \in \mathcal{V}_n^H$. Notice that

$$\nu(y_i) = \begin{cases} 1 & i \in J_1; \\ 2 & i \in J_3 \end{cases} \quad \text{and} \quad \mathcal{M}_0(V) = \text{span}_{\mathbb{F}}\{y_i y_j \mid (i, j) \in J_1 \times J_1 \cup J_3 \times J_3\}.$$

We may assume that h is a linear combination of monomials with value 2. When $h = y_i y_j$, $(i, j) \in (I_{01} \cup \bar{I}_{01}) \times J_3$, using the same method as in Case 1, we get $H_0 = \bar{M}$. When $h = \sum_{i \in I_{11}} a_i y_k y_i$, where $k \in J_3$, $a_i \in \mathbb{F}$, $a_j \neq 0$, we get $H_0 = \bar{M}$ if I_{01} is not empty or J_1 is single by a similar argument as in Case 1. Thus, it suffices to consider the condition that I_{01} is empty and J_1 is neither single nor twinned. For distinct $l, s, j \in I_{11}$, we have

$$\begin{aligned} y_k y_s &= (a_j)^{-1} \lambda_{k,s} [y_l y_s, [y_j y_l, h]] \in \bar{M}, \\ y_e y_k &= \lambda_{e,k} [y_k y_s, y_s y_e] \in \bar{M} \text{ for any } s \neq e \in I_{11}. \end{aligned}$$

For any $i \in I_{11}$, $f \in I_{03}$ and $t \in I_{13}$, we have

$$y_f y_i = \lambda_{f,i} [y_k y_i, y_{\tilde{k}} y_f] \in \bar{M} \text{ and } y_t y_i = \lambda_{t,i} [y_f y_i, y_{\tilde{f}} y_t] \in \bar{M}.$$

Thus, $H_0 = \bar{M}$.

Case 3. $V \in \mathcal{V}_0^H$. Notice that

$$\nu(y_i) = \begin{cases} 1 & i \in J_1; \\ 0 & i \in J_2; \\ \frac{1}{3} & i \in \bar{J}_2 \end{cases} \quad \text{and} \quad \mathcal{M}_0(V) = \text{span}_{\mathbb{F}} \{y_i y_j \mid (i, j) \in J_2 \times \mathbf{I} \cup J_1 \times J_1\}.$$

We have $H_0 = \bar{M}$ by the same method as in Cases 1 and 2.

Conversely, we consider the co-basis of $\mathcal{M}_0(V)$ in H_0 :

$$\{y_i y_j \mid (i, j) \in J_1 \times \bar{J}_2 \cup J_1 \times J_3 \cup \bar{J}_2 \times \bar{J}_2 \cup \bar{J}_2 \times J_3\}. \quad (4.27)$$

Notice that if $V \notin \mathcal{V}_i^H \cup \mathcal{V}_n^H \cup \mathcal{V}_0^H$, then $V \in \mathfrak{V}^H$ must satisfy one of the following conditions:

(i) None of J_1, J_2, J_3 is empty. In this case, we choose a monomial h of H_0 with $\nu(h) = 2$. Then there do not exist monomials with value $\frac{1}{3}$ in \bar{M} .

(ii) J_3 is twinned, i.e., $J_3 = \{j, \tilde{j}\}$, where $j \neq \tilde{j} \in \mathbf{I}_1$. In this case, let $h = y_i y_j$, where

$$i \in \begin{cases} \bar{J}_2, & J_1 \text{ is empty;} \\ J_1, & \text{otherwise.} \end{cases}$$

Then $y_i y_{\tilde{j}} \notin \bar{M}$.

(iii) J_1 is twinned, i.e., $J_1 = \{m+n-1, m+n\}$. In this case, let $h = y_k (y_{m+n-1} + \sqrt{-1} y_{m+n})$, where

$$k \in \begin{cases} \bar{J}_2, & J_3 \text{ is empty;} \\ J_3, & \text{otherwise.} \end{cases}$$

Then $y_k y_{m+n} \notin \bar{M}$.

Therefore, \bar{M} is a nontrivial subalgebra of H_0 strictly containing $\mathcal{M}_0(V)$ when (i), (ii) or (iii) holds, which implies that $\mathcal{M}_0(V)$ is not a maximal subalgebra of H_0 . \square

Proposition 4.8. *The subalgebra $\mathcal{M}(V)$ is maximal in H if and only if $V \in \mathcal{V}^H$.*

Proof. If $J_3 = \{m + n - d\}$, from Lemma 4.7(1), we know that

$$\mathcal{M}_i(V) = \text{span}_{\mathbb{F}}\{u \in H_i \mid \nu(u) = 1, 0\}, \quad i \geq -1,$$

which implies that

$$\text{alg}(\mathcal{M}(V) + \mathbb{F}y_{m+n-d}) \subset \text{span}_{\mathbb{F}}\{u \in H \mid \nu(u) = 1, 0\} + \mathbb{F}y_{m+n-d}.$$

Thus, $y_i, y_j y_{m+n-d} \notin \text{alg}(\mathcal{M}(V) + \mathbb{F}y_{m+n-d})$ if $\nu(y_i) = \frac{1}{3}$ or $\nu(y_j) = 1$, which contradicts the maximality of $\mathcal{M}(V)$.

If $J_3 = \{j, \tilde{j}\}$, where $j \neq \tilde{j} \in \mathbf{I}_1$, from Lemma 4.7(1), we know that

$$\mathcal{M}_i(V) = \text{span}_{\mathbb{F}}\{u \in H_i \mid \nu(u) = 1, 0, (\frac{1}{3})^k 4\}, \quad i \geq 0.$$

Then for any monomial $u \in \mathcal{M}_i(V)$, we have:

$$[y_j, u] = 0 \quad \text{or} \quad [y_j, u] = w y_j,$$

where $0 \neq w \in H_{i-1}$ with $D_{\tilde{j}}(w) = 0$, which implies that

$$y_{\tilde{j}} \notin \text{alg}(\mathcal{M}(V) + \mathbb{F}y_j).$$

This contradicts the maximality of $\mathcal{M}(V)$.

Conversely, let us prove the maximality of $\mathcal{M}(V)$. By definition (4.24), it is sufficient to show that $\overline{\mathcal{M}} = \text{alg}(\mathcal{M}(V) + \mathbb{F}h) = H$, where $h = y_i$, $i \in \bar{J}_2 \cup J_3$. Note that $\mathcal{M}_1(V) \neq 0$ for $|\mathbf{I}_0| \geq 2$. From Lemmas 2.1(2) and 3.11(1), it suffices to prove $H_{-1}, H_0 \subset \overline{\mathcal{M}}$. For $V \in \mathcal{V}^H$, we discuss the following cases:

Case 1. J_2 is not empty. When $i \in \bar{J}_2$, since

$$y_{\tilde{i}} \in V \quad \text{and} \quad y_j = \lambda_{i,j}[y_i, y_{\tilde{i}} y_j] \in \overline{\mathcal{M}} \quad \text{for} \quad \tilde{i} \neq j \in \mathbf{I},$$

we have $H_{-1} \subset \overline{\mathcal{M}}$. When $i \in J_3$, for all $j \in J_3$ with $j \neq i, \tilde{i}$, we have:

$$y_j = \lambda_{i,j}[y_i, y_{\tilde{i}} y_j] \quad \text{and} \quad y_{\tilde{i}} = \lambda_{\tilde{i},j}[y_j, y_{\tilde{i}} y_{\tilde{j}}].$$

Note that

$$y_l = \lambda_{l,i}[y_i, [y_{\tilde{i}}, y_i y_{\tilde{i}} y_l]] \in \overline{\mathcal{M}} \quad \text{for all} \quad l \in \bar{J}_2.$$

Thus we have $H_{-1} \subset \overline{\mathcal{M}}$. Note that for an arbitrary monomial $u \in H_0$, there exists $k \in \mathbf{I}$ such that $u y_k \neq 0$ and $\nu(u y_k) = 0$. Then we have

$$u = \lambda_{\tilde{k},k}[y_{\tilde{k}}, u y_k] \in \overline{\mathcal{M}},$$

which implies that $H_0 \subset \overline{\mathcal{M}}$. Thus, we have $\overline{\mathcal{M}} = H$.

Case 2. J_2 is empty. Obviously, J_1 and J_3 are not empty. Then we have $H_{-1}, H_0 \subset \overline{\mathcal{M}}$ by the same method as in Case 1. \square

To avoid confusion, we rewrite $\mathcal{M}_i^L(V)$ for $\mathcal{M}_i(V)$, $\mathcal{M}^L(V)$ for $\mathcal{M}(V)$, $L = H$ or K .

Lemma 4.9. Let $\gamma = \lfloor \frac{i+2}{2} \rfloor$ for $i > 0$. Put

$$\widetilde{\mathcal{M}}_j(V) = \begin{cases} 0, & j > \eta - 2; \\ \mathcal{M}_j^H(V), & j < \eta - 2; \end{cases} \quad \widetilde{\mathcal{M}}_{\eta-2}(V) = \begin{cases} 0, & V \text{ is nondegenerate} \\ & \text{and } J_3 \text{ is single;} \\ \mathbb{F}y^{(\pi)}y^\omega, & \text{otherwise,} \end{cases}$$

where $\pi = (p-1, \dots, p-1) \in \mathbb{N}^{2r}$, $\eta = 2r(p-1) + n$ and $\omega = \langle m+1, \dots, m+n \rangle$. Then

$$(1) \mathcal{M}_0^K(V) = \mathcal{M}_0^H(V) \oplus \mathbb{F}z.$$

(2) If V is not isotropic, for $i > 0$,

$$\mathcal{M}_i^K(V) = \widetilde{\mathcal{M}}_i(V) \oplus \widetilde{\mathcal{M}}_{i-2}(V)z \oplus \cdots \oplus \widetilde{\mathcal{M}}_{i-2\gamma}(V)z^\gamma.$$

(3) If V is isotropic, for $i > 0$,

$$\mathcal{M}_i^K(V) = \widetilde{\mathcal{M}}_i(V) \oplus \overline{H}_{i-2}z \oplus \cdots \oplus \overline{H}_{i-2\gamma}z^\gamma.$$

Proof. (1) It is obvious.

(2) Use induction on i . Clearly, $\widetilde{\mathcal{M}}_i(V) \subset \mathcal{M}_i^K(V)$.

“ \supset ”: For $gz^k \in \widetilde{\mathcal{M}}_{i-2k}(V)z^k$, $0 < k \leq \gamma$, we know that

$$[y_l, gz^k] = [y_l, g]z^k + y_lgz^{k-1}.$$

Note that

$$y_lg \in \widetilde{\mathcal{M}}_{i-2(k-1)-1}(V) \text{ for } \nu(y_l) = 1, 0.$$

By induction on $\text{zd}(g)$, we have:

$$y_lgz^{k-1}, [y_l, g]z^k \in \mathcal{M}_{i-1}^K(V).$$

Thus, $gz^k \in \mathcal{M}_i^K(V)$.

“ \subset ”: For any $u \in \mathcal{M}_i^K(V)$, by Lemma 3.11(2), we may assume that

$$u = u_i + u_{i-2}z + \cdots + u_{i-2\gamma}z^\gamma,$$

where $u_j \in \overline{H}_j$ for $i-2\gamma \leq j \leq i$. Note that $\mathcal{M}_{-2}^K(V) \neq 0$, since V is not isotropic. Then we have:

$$u_{i-2} + u_{i-4}z + \cdots + u_{i-2\gamma}z^{\gamma-1} = 2^{-1}[1, u] \in \mathcal{M}_{i-2}^K(V).$$

By induction, we have $u_j \in \widetilde{\mathcal{M}}_j(V)$ for $i-2\gamma \leq j \leq i-2$. Moreover,

$$u_{i-2}z + u_{i-4}z^2 + \cdots + u_{i-2\gamma}z^\gamma \in \mathcal{M}_i^K(V).$$

Consequently, $u_i \in \widetilde{\mathcal{M}}_i(V)$.

(3) When V is isotropic, note that $\nu(y_k) = 0$ for all $y_k \in V$. The remaining discussion is analogous to that of the condition (2). \square

Proposition 4.10. The subalgebra $\mathcal{M}^K(V)$ is maximal in K if and only if $V \in \mathcal{V}^K$ when V is neither nondegenerate nor isotropic; $V \in \mathcal{W}^K$, otherwise.

Proof. The proof of the necessity is similar to the one of Proposition 4.8. We only consider the sufficiency. For any $u \in K$, $u \notin \mathcal{M}^K(V)$, put $\overline{M} = \text{alg}(\mathcal{M}^K(V) + \mathbb{F}u)$. Then there exist $v_1, \dots, v_i \in V$ such that

$$0 \neq h = [v_1, [\dots, [v_i, u] \dots]] \in \overline{M} \cap K_{-1} \quad \text{and} \quad h \notin V.$$

When J_3 is neither single nor twinned, by Proposition 4.8, we have $H \subset \overline{M}$. When J_3 is single and V is isotropic, we may assume that $h = y_i$, $i \in \bar{J}_2 \cup \{m+n-d\}$. If $i \in \bar{J}_2$, for any $j, k \in \bar{J}_2$, we obtain that

$$y_{m+n-d} = \lambda_{i,\tilde{i}}[y_{\tilde{i}}, [y_i, y_{m+n-d}z]], \quad y_{m+n-d}y_jy_k = \lambda_{j,k}[y_{m+n-d}, y_jy_kz]$$

are in \overline{M} from Lemma 4.9(3). Moreover,

$$y_jy_k = -[y_{m+n-d}, y_{m+n-d}y_jy_k], \quad y_{m+n-d}y_k = \lambda_{m+n-d,k}[y_{\tilde{j}}, y_{m+n-d}y_jy_k]$$

are in \overline{M} . Keeping in mind the co-basis (4.27), we have $H_0 \subset \overline{M}$, which also holds when J_3 is single and V is nondegenerate. From the irreducibility of K_{-1} , K_{10} and K_{11} , as well as Lemma 4.9, we obtain that $\mathcal{M}^K(V)$ is maximal in K . \square

In the same way as in Proposition 4.10, one may check the following proposition.

Proposition 4.11. *If V is isotropic, the subalgebra $\mathcal{M}^K(1, V)$ is maximal in K if and only if $V \in \mathcal{V}^K$.*

Convention 4.12. *For simplicity, put $\mathcal{O}_X = \text{span}_{\mathbb{F}}\{y_{i_1} \cdots y_{i_s} \mid i_1, \dots, i_s \in X, s \geq 1\}$, $\mathcal{Q}_X = \text{span}_{\mathbb{F}}\{y_i \mid i \in X\}$ and $\mathcal{Y}_X^+ = \mathcal{Y}_X \oplus \mathbb{F} \cdot 1$, where X is a subset of \mathbf{I} and $\mathcal{Y} = \mathcal{O}$ or \mathcal{Q} .*

Proof of Theorem 4.2 For (1) and (1'), the proofs follow from Lemma 4.3(3), Propositions 4.8, 4.10 and 4.11.

For H , from Lemma 4.7(1) and (1), we obtain that

$$\dim \mathcal{M}^H(V) = \dim H - \dim(\mathcal{O}_{J_1}^+ \mathcal{O}_{\bar{J}_2} \oplus \mathcal{O}_{J_1 \cup \bar{J}_2}^+ \mathcal{Q}_{J_3}).$$

For K , from Lemma 4.9 and (1'), we obtain that

$$\begin{aligned} \dim \mathcal{M}^K(1, V) &= p(\dim \mathcal{M}^H(V) + 2); \\ \dim \mathcal{M}^K(V) &= \begin{cases} \dim \mathcal{M}^H(V) + 1 + (p-1)(\dim H + 2), & V \in \mathcal{W}^K \text{ is isotropic;} \\ p(\dim \mathcal{M}^H(V) + 2), & \text{otherwise.} \end{cases} \end{aligned}$$

By a standard and direct computation we get the formulas (4) and (4'). Noting that $\dim \mathcal{M}_0(V) = \dim \mathcal{M}_0(V')$ if $\mathcal{M}(V) \cong \mathcal{M}(V')$ and using the same method as in Theorem 4.1(2), (2) and (2') hold. From (1), (1') and (2), (2'), we obtain that (3) and (3') hold. \square

5. MGS of Type (III)

Suppose $L = W, S, H$ or K . Recall that an MGS of type (III) of L , M , satisfies the condition

$$M_{-1} = L_{-1} \quad \text{and} \quad M_0 \neq L_0.$$

Let \mathfrak{G}_0 be a nontrivial subalgebra of L_0 . Define a graded subspace of L as follows:

$$\mathcal{M}(L_{-1}, \mathfrak{G}_0) = \oplus_{i \geq -2} \mathcal{M}_i(L_{-1}, \mathfrak{G}_0),$$

where

$$\begin{aligned} \mathcal{M}_{-i}(L_{-1}, \mathfrak{G}_0) &= L_{-i}, \quad i < 0; \quad \mathcal{M}_0(L_{-1}, \mathfrak{G}_0) = \mathfrak{G}_0; \\ \mathcal{M}_i(L_{-1}, \mathfrak{G}_0) &= \{u \in L_i \mid [L_{-1}, u] \subset \mathcal{M}_{i-1}(L_{-1}, \mathfrak{G}_0)\} \quad \text{for } i > 0. \end{aligned} \quad (5.28)$$

It is easy to see that $\mathcal{M}(L_{-1}, \mathfrak{G}_0)$ is a graded subalgebra satisfying the condition (III). We call \mathfrak{G} a *maximal R-subalgebra* (resp. *maximal S-subalgebra*) of L if \mathfrak{G} is maximal reducible (resp. irreducible) graded and satisfies the condition (III). All the MGS of type (III) can be split into the disjoint union of maximal R-subalgebras and maximal S-subalgebras.

Theorem 5.1. *Suppose $L = W$ or S .*

- (1) *All maximal R-subalgebras of L are precisely:*

$$\{\mathcal{M}(L_{-1}, \mathcal{M}_0(V)) \mid V \in \mathfrak{V}^L\}.$$

- (2) *For any $V, V' \in \mathfrak{V}^L$,*

$$\mathcal{M}(L_{-1}, \mathcal{M}_0(V)) \stackrel{\text{algebra}}{\cong} \mathcal{M}(L_{-1}, \mathcal{M}_0(V')) \iff V \cong V'.$$

- (3) *L has exactly $(m+1)(n+1)-2$ isomorphism classes of maximal R-subalgebras.*

- (4) *Suppose $V \in \mathfrak{V}^L$ with $\text{superdim} V = (k, l)$, then*

$$\begin{aligned} &\dim \mathcal{M}(L_{-1}, \mathcal{M}_0(V)) \\ &= \begin{cases} 2^{n-l} p^{m-k} (m+n-k-l) + 2^n p^m (k+l), & L = W \\ 2^{n-l} p^{m-k} (m+n) + 2^n p^m (k+l-1) - k-1, & L = S. \end{cases} \end{aligned}$$

Let $L = H$ or K and put $V^\perp = \{u \in L_{-1} \mid \beta(u, V) = 0\}$ for $V \in \mathfrak{V}^L$. Recall definitions (2.16)–(2.18) and (4.26).

Theorem 5.2. *All maximal R-subalgebras of H and K are characterized as follows:*

For H ,

- (1) *All maximal R-subalgebras of H are precisely:*

$$\{\mathcal{M}(H_{-1}, \mathcal{M}_0(V)) \mid V \in \mathcal{V}_{\mathfrak{n}}^H \cup \mathcal{V}_{\mathfrak{i}}^H\}.$$

- (2) *Suppose $V, V' \in \mathcal{V}_{\mathfrak{n}}^H \cup \mathcal{V}_{\mathfrak{i}}^H$. Then*

$$\mathcal{M}(H_{-1}, \mathcal{M}_0(V)) \stackrel{\text{algebra}}{\cong} \mathcal{M}(H_{-1}, \mathcal{M}_0(V'))$$

if and only if one of the following conditions holds.

(i) $V \cong V'$.

(ii) $V^\perp \cong V'^\perp$ when V and V' are both nondegenerate.

(3) H has exactly $\phi(r, n)$ isomorphism classes of maximal R -subalgebras, where

$$\phi(r, n) = \begin{cases} 2^{-1}(nr + 3n + 2r - 2) + \lfloor \frac{r}{2} \rfloor (n + 1), & n \text{ is even;} \\ 2^{-1}(nr + 3n + r - 1) + \lfloor \frac{r}{2} \rfloor (n + 1), & n \text{ is odd.} \end{cases}$$

(4) Suppose $V \in \mathcal{V}_n^H \cup \mathcal{V}_i^H$ with $\beta\text{-dim}V = (a, b, c, d)$, then

$$\dim \mathcal{M}(H_{-1}, \mathcal{M}(V)) = \begin{cases} p^{2a}2^d + p^{2(r-a)}2^{(n-d)} - 2, & V \in \mathcal{V}_n^H; \\ p^{m-b}2^{n-c} + (b+c)p^b2^c - 1, & V \in \mathcal{V}_i^H. \end{cases}$$

For K ,

(1') All maxima R -subalgebras of K are precisely:

$$\{\mathcal{M}(K_{-1}, \mathcal{M}_0(V)) \mid V \in \mathcal{V}_i^K\}.$$

(2') Suppose $V, V' \in \mathcal{V}_i^K$. Then

$$\mathcal{M}(K_{-1}, \mathcal{M}_0(V)) \stackrel{\text{algebra}}{\cong} \mathcal{M}(K_{-1}, \mathcal{M}_0(V')) \iff V \cong V'.$$

(3') K has exactly $\phi(r, n)$ isomorphism classes of maximal R -subalgebras, where

$$\phi(r, n) = \begin{cases} 2^{-1}(rn + n + 2r - 2), & n \text{ is even;} \\ 2^{-1}(rn + n + r - 1), & n \text{ is odd.} \end{cases}$$

(4') Suppose $V \in \mathcal{V}_i^K$ with $\beta\text{-dim}V = (0, b, c, 0)$, then

$$\dim \mathcal{M}(K_{-1}, \mathcal{M}_0(V)) = \begin{cases} p^{b+1}2^c(b+c+1), & J_3 \text{ is empty;} \\ p^b2^c(p^{m-2b}2^{n-2c} + b+c), & \text{otherwise.} \end{cases}$$

Unfortunately, for maximal S -subalgebras, we have not obtained a similar description as for the maximal graded subalgebras of type (I) or (II) as well as for the maximal R -subalgebras. However, the classification of maximal S -subalgebras of L can be reduced to that of the maximal irreducible subalgebras of the classical Lie superalgebras (see Lemma 2.1(3)).

Theorem 5.3. Suppose $L = W, S, H$ or K . All maximal S -subalgebra of L are characterized as follows:

- (1) Every maximal S -subalgebra of L is of the form $\mathcal{M}(L_{-1}, \mathfrak{G}_0)$, where \mathfrak{G}_0 is a maximal irreducible subalgebra of L_0 .
- (2) Suppose \mathfrak{G}_0 is a maximal irreducible subalgebra of L_0 .

(a) For $L = W$, $\mathcal{M}(W_{-1}, \mathfrak{G}_0)$ is maximal in W if and only if

$$\text{div}(\mathcal{M}_1(W_{-1}, \mathfrak{G}_0)) \neq 0.$$

(b) For $L = S$ or H , $\mathcal{M}(L_{-1}, \mathfrak{G}_0)$ is maximal in L if and only if

$$\mathcal{M}_1(L_{-1}, \mathfrak{G}_0) \neq 0.$$

- (c) For $L = K$, $\mathcal{M}(K_{-1}, \mathfrak{G}_0)$ is a maximal in K if and only if there exists $u \in \mathcal{M}_1(K_{-1}, \mathfrak{G}_0)$ satisfying
- $$[1, u] \neq 0.$$

Let $L = W, S, H$ or K . As in the case of modular Lie algebras [16], it is easy to show the following lemmas.

Lemma 5.4. *Let \mathfrak{G}_0 be a nontrivial subalgebra of L_0 . If Φ is a \mathbb{Z} -homogeneous automorphism of L . Then*

$$\Phi(\mathcal{M}_i(L_{-1}, \mathfrak{G}_0)) = \mathcal{M}_i(L_{-1}, \Phi(\mathfrak{G}_0)) \text{ for all } i \geq -2.$$

Moreover,

$$\Phi(\mathcal{M}(L_{-1}, \mathfrak{G}_0)) = \mathcal{M}(L_{-1}, \Phi(\mathfrak{G}_0)).$$

Lemma 5.5. *Let $M = L_{-1} + M_0 + M_1 + M_2 + \cdots$ be any MGS of L . Then M_0 is maximal in L_0 unless $M_0 = L_0$.*

Lemma 5.6. *If M is an MGS of type (III) of L then M_0 is maximal in L_0 and $M = \mathcal{M}(L_{-1}, M_0)$.*

Lemma 5.7. *If \mathfrak{G}_0 is a maximal reducible subalgebra of L_0 then there exists a $V \in \mathfrak{V}^L$ such that $\mathfrak{G}_0 = \mathcal{M}_0(V)$ and $\mathcal{M}_i(L_{-1}, \mathfrak{G}_0) \subset \mathcal{M}_i(V)$ for $i \geq 0$. Conversely, $\mathcal{M}_0(V)$ is a reducible maximal subalgebra of L_0 if $V \in \mathfrak{V}^L$ when $L = W$ or S ; if $V \in \mathcal{V}_n^L \cup \mathcal{V}_i^L \cup \mathcal{V}_0^L$ when $L = H$ or K .*

Proof. Since \mathfrak{G}_0 is reducible, L_{-1} has a nontrivial \mathfrak{G}_0 -submodule V . From definition (4.24) and the maximality of \mathfrak{G}_0 , we have $\mathfrak{G}_0 = \mathcal{M}_0(V)$. From definitions (4.24) and (5.28), we have $\mathcal{M}_i(L_{-1}, \mathfrak{G}_0) \subset \mathcal{M}_i(V)$ for $i \geq 0$. The second statement follows immediately from Lemmas 4.5(1) and 4.7(2). \square

Remark 5.8. *In view of Lemmas 2.2, 5.4 and 5.7, if \mathfrak{G}_0 is a maximal reducible subalgebra of L_0 , we may assume that V is a standard element in \mathfrak{V}^L [see (2.15), 2.18)] such that $\mathfrak{G}_0 = \mathcal{M}_0(V)$.*

Proposition 5.9. *Suppose $L = W$ or S . $\mathcal{M}(L_{-1}, \mathfrak{G}_0)$ is a maximal R -subalgebra, if \mathfrak{G}_0 is a maximal reducible subalgebra of L_0 .*

Proof. Let us show that $M = \mathcal{M}(L_{-1}, \mathfrak{G}_0)$ is maximal. Assume that \overline{M} is a maximal graded subalgebra containing M . Clearly, $\overline{M}_{-1} = L_{-1}$. Since \mathfrak{G}_0 is a maximal subalgebra of L_0 , we have $\overline{M}_0 = \mathfrak{G}_0$ or L_0 . If $\overline{M}_0 = \mathfrak{G}_0$ then \overline{M} is an MGS of type (III). By Lemma 5.6,

$$\overline{M} = \mathcal{M}(L_{-1}, \mathfrak{G}_0) = M$$

and we are done. Let us consider the remaining case; $\overline{M}_0 = L_0$. Clearly, \overline{M} is an MGS of type (I) and by Theorem 3.1,

$$M_1 \subset \overline{M}_1 = W'_1, W''_1, S''_1, \text{ or } \{0\}. \quad (5.29)$$

On the other hand, by Lemma 5.7, there exists a $V \in \mathfrak{V}^L$ such that $\mathfrak{G}_0 = \mathcal{M}_0(V)$. Assume that V has a standard basis:

$$(\partial_1, \dots, \partial_k \mid \partial_{m+1}, \dots, \partial_{m+l}).$$

Hence $\mathfrak{G}_0 = \mathcal{M}_0(V)$ has a standard co-basis (4.25) in W_0 :

$$\mathcal{A}_3 = \{x_i \partial_j \mid i \in \mathbf{I}(k, l), j \in \bar{\mathbf{I}}(k, l)\}.$$

To reach a contradiction, in view of (5.29), it is sufficient to find an element belonging to M_1 but not W', W'' for W , but not S'' or $\{0\}$ for S . For $L = W$, $x_j x_i \partial_i$ with $i \in \mathbf{I}(k, l)$ and an arbitrarily chosen j is a desired element. Here we have used the fact that both $|\mathbf{I}(k, l)| \geq 1$ and $|\bar{\mathbf{I}}(k, l)| \geq 1$, since $V \in \mathfrak{V}^W$. For $L = S$, pick distinct i, j, r with $i \in \mathbf{I}(k, l)$ and with j, r chosen arbitrarily. Here note that the general assumption ensures $|\mathbf{I}| \geq 4$. Then $x_j x_r \partial_i \in S_1$ is a desired candidate for S . The proof is complete. \square

Proof of Theorem 5.1 (1) This follows from Lemmas 5.5, 5.6, 5.7 and Proposition 5.9.

(2) One implication is obvious. Suppose Φ is an isomorphism of $\mathcal{M}(L_{-1}, \mathcal{M}_0(V))$ onto $\mathcal{M}(L_{-1}, \mathcal{M}_0(V'))$. Consequently, $\Phi(L_{-1}) = L_{-1}$ and $\Phi(\mathcal{M}_0(V)) = \mathcal{M}_0(V')$. A standard verification shows that $\Phi(\mathcal{M}_0(V)) = \mathcal{M}_0(\Phi(V))$. By Lemma 4.5(2), we have $\Phi(V) = V'$.

(3) This is a direct consequence of (2).

(4) Suppose V is a standard element in \mathfrak{V}^L . Then $\mathcal{M}(W_{-1}, \mathcal{M}_0(V))$ has a standard \mathbb{F} -basis

$$\begin{aligned} & \{x^{(\alpha)} x^u \partial_i \mid \alpha \in \mathbf{A}(m), u \in \mathbf{B}(n); i \in \mathbf{I}(k, l)\} \\ & \cup \{x^{(\alpha)} x^u \partial_i \mid \alpha_1 = \cdots = \alpha_k = 0, u \in \overline{m+l+1, m+n}; i \in \bar{\mathbf{I}}(k, l)\}. \end{aligned}$$

Thus, we have:

$$\dim \mathcal{M}(W_{-1}, \mathcal{M}_0(V)) = 2^{n-l} p^{m-k} (m+n-k-l) + 2^n p^m (k+l).$$

Note that $\bar{S} = S \oplus \sum_{i \in \mathbf{I}_0} x^{(\pi-(p-1)\varepsilon_i)} x^\omega \partial_i$, where $\pi = (p-1, \dots, p-1) \in \mathbb{N}^m$ and $\omega = \langle m+1, \dots, m+n \rangle$. Then we have:

$$\dim \mathcal{M}(S_{-1}, \mathcal{M}_0(V)) = 2^{n-l} p^{m-k} (m+n) + 2^n p^m (k+l-1) - k-1.$$

\square

We call $\mathfrak{G}_0 = \mathcal{M}_0(V)$ is *degenerate* if $V \in \mathcal{V}_1^L \cup \mathcal{V}_0^L$.

Proposition 5.10. *Let \mathfrak{G}_0 be a maximal reducible subalgebra of H_0 or K_0 .*

(1) $\mathcal{M}(H_{-1}, \mathfrak{G}_0)$ is maximal in H .

(2) $\mathcal{M}(K_{-1}, \mathfrak{G}_0)$ is maximal in K if and only if \mathfrak{G}_0 is degenerate.

Proof. For any $0 \neq h \in L$, $h \notin \mathcal{M}(L_{-1}, \mathfrak{G}_0)$, put $\bar{M} = \text{alg}(\mathcal{M}(L_{-1}, \mathfrak{G}_0) + \mathbb{F}h)$. By the maximality of \mathfrak{G}_0 , we have $L_0 \subset \bar{M}$. For H , choose $k \in I_{0i}$ if I_{0i} is not empty where $i = 1, 2$ or 3 . It follows that $y_k^3 \in \mathcal{M}_1(H_{-1}, \mathfrak{G}_0)$. For K , suppose \mathfrak{G}_0 is degenerate. Using the same method as for H , we can find $0 \neq v_i \in \bar{M} \cap K_{1i}$, where $i = 0, 1$. From Lemmas 2.1(2) and 3.11, we have $\bar{M} = L$.

It remains to show that \mathfrak{G}_0 is degenerate if $\mathcal{M}_1(K_{-1}, \mathfrak{G}_0)$ is maximal. Assume on the contrary that $V_{\mathfrak{G}_0} \in \mathcal{V}_n^K$ is a nondegenerate irreducible \mathfrak{G}_0 -module. For any $u \in \mathcal{M}_1(K_{-1}, \mathfrak{G}_0)$, by Lemmas 4.9(2) and 5.7, we may assume that

$$u = f_{-1}z + f_1, \text{ where } f_{-1} \in V_{\mathfrak{G}_0} \text{ and } f_1 \in \widetilde{\mathcal{M}}_1(V_{\mathfrak{G}_0}).$$

Note that f_{-1} is a linear combination of monomials with value 1. Let $f_1 = f^1 + f^4 + f^8$ where f^i is a linear combination of monomials with value i , $i = 1, 4$ or 8 . We claim that $f_{-1} = 0$. Indeed, for any $y_i \in K_{-1}$ with value 2, we have

$$\sigma(i)(-1)^i(D_{\bar{i}}(f_1) + D_{\bar{i}}(f_{-1})z) + y_i f_{-1} = [y_i, u] \in \mathcal{M}_0(V_{\mathfrak{G}_0}) = \mathfrak{G}_0,$$

which implies that $f_{-1} = 0$ when J_3 is single. Otherwise, the following equation holds:

$$\sigma(i)(-1)^i D_{\bar{i}}(f^4) = -y_i f_{-1}. \quad (5.30)$$

Then there exists $g_1 \in K_1$ with $D_{\bar{i}}(g_1) = 0$ satisfying

$$\sigma(i)(-1)^i f^4 = -y_i y_i f_{-1} + g_1. \quad (5.31)$$

By equations (5.30) and (5.31), we have $D_{\bar{i}}(g_1) = (-1)^i 2y_i f_{-1}$ which contradicts $D_{\bar{i}}(g_1) = 0$ if $f_{-1} \neq 0$. Consequently, $\mathcal{M}_1(K_{-1}, \mathfrak{G}_0) \subset K_{10}$. Using induction on k and the transitivity of K , we have $\mathcal{M}_k(K_{-1}, \mathfrak{G}_0) \subset K_{k0}$ for $k > 0$. It follows that $\mathcal{M}(K_{-1}, \mathfrak{G}_0)$ is strictly contained in $K_{-2} + K_{-1} + K_0 + \sum_{i=1}^{2r(p-1)+n} K_{i0}$. The latter is a maximal graded subalgebra (see Theorem 3.1(4)). This contradicts the maximality of $\mathcal{M}(K_{-1}, \mathfrak{G}_0)$. The proof is complete. \square

Lemma 5.11. *Suppose $L = H$ or K .*

(1) *Suppose $V \in \mathcal{V}_0^L$. Then V contains a subspace $V' \in \mathcal{V}_1^L$ such that*

$$\mathcal{M}(L_{-1}, \mathcal{M}_0(V)) = \mathcal{M}(L_{-1}, \mathcal{M}_0(V')).$$

(1) *If $V, V' \in \mathcal{V}_n^L \cup \mathcal{V}_i^L$, then*

$$\mathcal{M}_0(V) = \mathcal{M}_0(V')$$

if and only if one of the following conditions holds.

(i) $V = V'$.

(ii) $V^\perp = V'$ when V and V' are nondegenerate.

Proof. For (1), we may assume that $V = \text{span}_{\mathbb{F}}\{y_i \mid i \in J_1 \cup J_2\}$. Then $V' = \text{span}_{\mathbb{F}}\{y_i \mid i \in J_2\}$ is desired. For (2), by a similar argument as in Lemma 4.5(2), we get the desired conclusion. \square

Lemma 5.12. *The following statements hold.*

(1) *If $V \in \mathcal{V}_n^H$, then $\mathcal{M}(H_{-1}, \mathcal{M}_0(V)) = \mathcal{O}_{J_1} \oplus \mathcal{O}_{J_3}$.*

(2) *If $V \in \mathcal{V}_i^H$, then $\mathcal{M}(H_{-1}, \mathcal{M}_0(V)) = \mathcal{O}_{J_2 \cup J_3} \oplus \mathcal{O}_{J_2}^+ \mathcal{Q}_{\bar{J}_2}$.*

Proof. (1) For $V \in \mathcal{V}_n^H$, a direct computation shows that

$$\mathcal{M}_i(H_{-1}, \mathcal{M}_0(V)) = \text{span}_{\mathbb{F}}\{u \in H_i \mid u \text{ is a monomial with } \nu(u) = 1, 2^{i+2}\}.$$

(2) For $V \in \mathcal{V}_i^H$, using induction on i , we obtain that $\mathcal{M}_i(H_{-1}, \mathcal{M}_0(V))$ is spanned by monomials in H as follows:

- (a) $u_1 u_2 \in H_i$, where u_1 is a monomial with the variables of value 0 and u_2 is a monomial with the variables of value 2.
- (b) $y_j u_3 \in H_i$, where $j \in \bar{J}_2$ and u_3 is a monomial with the variables of value 0.

Then, the conclusions hold. \square

For $u \in K$, put $Z(u) = i$ if $(ad1)^{i+1}u = 0$ and $(ad1)^i u \neq 0$.

Lemma 5.13. *Suppose $V \in \mathcal{V}_i^K$. For any element $u \in K$ with $[1, u] \neq 0$, $u \in \mathcal{M}(K_{-1}, \mathcal{M}_0(V))$ if and only if u is a linear combination of elements of the form $f(z+x)^j + g$, where $g \in \mathcal{O}_{J_2 \cup J_3}^+ \oplus \mathcal{O}_{J_2}^+ \mathcal{Q}_{\bar{J}_2}$,*

$$f \in \begin{cases} \mathcal{O}_{J_2}^+ \mathcal{Q}_{\bar{J}_2}^+, & V \in \mathcal{V}_i^K \text{ satisfying } J_3 \text{ is empty;} \\ \mathcal{O}_{J_2 \cup J_3}^+, & V \in \mathcal{V}_i^K \text{ satisfying } J_3 \text{ is not empty,} \end{cases}$$

$x = \sum_{i \in J_2} y_i y_{\bar{i}}$ and $0 < j < p$.

Proof. Let $\mathfrak{G}_0 = \mathcal{M}_0(V)$. Notice that g , x and $z+x \in \mathcal{M}(K_{-1}, \mathfrak{G}_0)$. Firstly, for any $i \in \mathbf{I}$, one computes

$$\begin{aligned} [y_i, z+x] &\in \mathcal{O}_{J_2 \cup J_3}^+, \\ f[y_i, z+x] &\in \mathcal{O}_{J_2 \cup J_3} + \mathcal{O}_{J_2} \mathcal{Q}_{\bar{J}_2}, \\ [y_i, f] &\in \begin{cases} \mathcal{O}_{J_2}^+ \mathcal{Q}_{\bar{J}_2}^+, & V \in \mathcal{V}_i^K \text{ satisfying } J_3 \text{ is empty;} \\ \mathcal{O}_{J_2 \cup J_3}^+, & V \in \mathcal{V}_i^K \text{ satisfying } J_3 \text{ is not empty.} \end{cases} \end{aligned}$$

Using induction on $\text{zd}(f)$ and j , respectively, we have

$$f(z+x), (z+x)^j \in \mathcal{M}(K_{-1}, \mathfrak{G}_0).$$

Furthermore, $f(z+x)^j \in \mathcal{M}(K_{-1}, \mathfrak{G}_0)$.

Conversely, let us use induction on $Z(u)$. When $Z(u) = 1$, we consider the following cases.

Case 1. $u \in \mathcal{M}_0(K_{-1}, \mathfrak{G}_0)$. By Lemmas 4.9(1) and 5.7, we may assume that $u = z + u_0$, where $u_0 \in \mathfrak{G}_0 \cap H_0$, which means that $u_0 \in \mathcal{O}_{J_2 \cup J_3} + \mathcal{O}_{J_2} \mathcal{Q}_{\bar{J}_2}$. Thus, $u = z + x + (u_0 - x)$ is desired.

Case 2. $u \in \mathcal{M}_1(K_{-1}, \mathfrak{G}_0)$. From Remark 2.3, we may assume that $u = y_t z + u_1$, where $u_1 \in H_1$ and $t \in \mathbf{I}$. Notice that, when $y_t(z+x) \in \mathcal{M}(K_{-1}, \mathfrak{G}_0)$,

$$u_1 - y_t x = u - y_t(z+x) \in \mathcal{M}(K_{-1}, \mathfrak{G}_0) \cap H,$$

which follows that $u = y_t(z+x) + (y_t x - u_1)$ is desired. Thus, by the necessity of this lemma, it is sufficient to consider the case of $t \in \bar{J}_2$ when J_3 is not empty. From Lemmas 4.9(3) and 5.7, we may assume that

$$u_1 = h^{(\frac{1}{3}, 2, 2)} + h^{(0, \frac{1}{3}, \frac{1}{3})} + h^{(0, \frac{1}{3}, 2)} + h,$$

where

$$h^{(\alpha, \beta, \gamma)} = \text{span}_{\mathbb{F}}\{y_i y_j y_k \mid \nu(y_i) = \alpha, \nu(y_j) = \beta, \nu(y_k) = \gamma\},$$

$h \in \mathcal{M}_1(K_{-1}, \mathfrak{G}_0) \cap H$. For any $y_l \in K_{-1}$, we have:

$$\sigma(l)(-1)^l(D_{\tilde{l}}(y_t)z + D_{\tilde{l}}u_1) + y_ly_t = [y_l, u] \in \mathfrak{G}_0,$$

which means that

$$\sigma(l)(-1)^l D_{\tilde{l}}u_1 + y_ly_t \in \mathfrak{G}_0. \quad (5.32)$$

When $\nu(y_l) = 2$, from equation (5.32) we have:

$$\sigma(l)(-1)^l D_{\tilde{l}}h^{(\frac{1}{3}, 2, 2)} + y_ly_t \in \mathfrak{G}_0,$$

which is a linear combination of elements with value $\frac{2}{3}$. It follows that

$$\sigma(l)(-1)^l D_{\tilde{l}}h^{(\frac{1}{3}, 2, 2)} + y_ly_t = 0.$$

When J_3 is single, we have $y_ly_t = 0$, a contradiction. When J_3 is not single, there exist distinct $k, \tilde{k} \in J_3$ such that

$$h^{(\frac{1}{3}, 2, 2)} = -\sigma(k)(-1)^k y_{\tilde{k}}y_k y_t + h', \quad (5.33)$$

where $D_{\tilde{k}}h' = 0$ and

$$\sigma(\tilde{k})(-1)^k D_k(h^{(\frac{1}{3}, 2, 2)}) + y_{\tilde{k}}y_t = 0.$$

From equation (5.33), we have

$$D_k(h') = D_k(h^{(\frac{1}{3}, 2, 2)}) + \sigma(k)y_{\tilde{k}}y_t = 2\sigma(k)y_{\tilde{k}}y_t,$$

which contradicts $D_{\tilde{k}}h' = 0$. Thus, an element of the form $y_t z + u_1$, $t \in \bar{J}_2$ is not in $\mathcal{M}_1(K_{-1}, \mathfrak{G}_0)$ when J_3 is not empty.

Case 3. $u \in \mathcal{M}_i(K_{-1}, \mathfrak{G}_0)$ for $i > 1$. We may assume that

$$u = g_{i-2}z + g_i, \quad g_j \in \overline{H}_j, \quad j = i-2, i.$$

Note that the elements of the form $h_2 z + h$ are not in $\mathcal{M}(K_{-1}, \mathfrak{G}_0)$, where h_2 is a linear combination of monomials with value $\frac{1}{9}$. By induction on i , we obtain that g_{i-2} is in $\mathcal{O}_{J_2} \mathcal{Q}_{\bar{J}_2}$ if J_3 is empty; in $\mathcal{O}_{J_2 \cup J_3}$, otherwise. Thus, $g_{i-2}(z+x) \in \mathcal{M}_i(K_{-1}, \mathfrak{G}_0)$. Moreover, $g_i - g_{i-2}x \in \mathcal{M}_i(K_{-1}, \mathfrak{G}_0) \cap \overline{H}$. Then $u = g_{i-2}(z+x) + (g_i - g_{i-2}x)$ is desired.

When $Z(u) = k > 1$, suppose

$$u = u_k z^k + u_{k-1} z^{k-1} + \cdots + u_1 z + u_0, \quad u_j \in \overline{H}, \quad j = 0, \dots, k.$$

Obviously,

$$u_k z + u_{k-1} = 2^{(1-k)}(\text{ad } 1)^{k-1}(u) \in \mathcal{M}(K_{-1}, \mathfrak{G}_0).$$

Thus, u_k is in $\mathcal{O}_{J_2}^+ \mathcal{Q}_{\bar{J}_2}^+$ when J_3 is empty; in $\mathcal{O}_{J_2 \cup J_3}^+$, otherwise. Consequently, $u_k(z+x)^k \in \mathcal{M}(K_{-1}, \mathfrak{G}_0)$. Thus,

$$v = u - u_k(z+x)^k \in \mathcal{M}(K_{-1}, \mathfrak{G}_0)$$

and $Z(v) < k$. By the inductive hypothesis, v is a linear combination of the desired form. So is u . The proof is complete. \square

Proof of Theorem 5.2. For (2) and (2'), sufficiency is obvious. For necessity, suppose Φ is an isomorphism of $\mathcal{M}(L_{-1}, \mathcal{M}_0(V))$ onto $\mathcal{M}(L_{-1}, \mathcal{M}_0(V'))$. Then, $\Phi(L_{-1}) = L_{-1}$ and $\Phi(\mathcal{M}_0(V)) = \mathcal{M}_0(V')$, which implies that $\dim \mathcal{M}_0(V) = \dim \mathcal{M}_0(V')$. It follows that V and V' are both nondegenerate or are both isotropic. Notice that $\Phi(\mathcal{M}_0(V)) \subset \mathcal{M}_0(\Phi(V))$. For the maximality of $\mathcal{M}_0(V')$, we have $\mathcal{M}_0(V') = \mathcal{M}_0(\Phi(V))$. By virtue of Lemma 5.11(2), we have $V' = \Phi(V)$ or $V' = \Phi(V)^\perp$ when V' and $\Phi(V)$ are both nondegenerate. Thus, we have $\dim V = \dim V'$ or $\dim V = m + n - \dim V'$. We can obtain the desired conclusions by a direct computation. (3) and (3') are direct consequences of (2) and (2').

The remaining statements hold from Lemmas 5.5, 5.6, 5.11, 5.12, 5.13 and Proposition 5.10. \square

Finally, we consider the maximal S-subalgebras of L , where $L = W, S, H$ or K . As in the case of modular Lie algebras [16], it is easy to show the following:

Lemma 5.14. *Suppose \mathfrak{G}_0 is a maximal irreducible subalgebra of L_0 . The subalgebra $\mathcal{M}(L_{-1}, \mathfrak{G}_0)$ is not maximal in L if $\mathcal{M}_1(L_{-1}, \mathfrak{G}_0) = 0$.*

Proof of Theorem 5.3. (1) This is nothing but Lemma 5.6.

(2) Let \mathfrak{G}_0 be a maximal irreducible subalgebra of L_0 .

(a) Suppose $L = W$ and $\mathcal{M}(W_{-1}, \mathfrak{G}_0)$ is maximal in W . Assume on the contrary that $\dim(\mathcal{M}_1(W_{-1}, \mathfrak{G}_0)) = 0$. By induction on i , one has $\mathcal{M}_i(W_{-1}, \mathfrak{G}_0) \subset W'_i$ for all $i \geq 1$. Since \mathfrak{G}_0 is a nontrivial subalgebra of W_0 , we have

$$\mathcal{M}(W_{-1}, \mathfrak{G}_0) \subsetneq W_{-1} + W_0 + W'_1 + W'_2 + \cdots$$

By Theorem 3.1, the latter is an MGS of W . This contradicts the maximality of $\mathcal{M}(W_{-1}, \mathfrak{G}_0)$.

Conversely, to show the maximality of $\mathcal{M}(W_{-1}, \mathfrak{G}_0)$, assume that M is an MGS strictly containing $\mathcal{M}(W_{-1}, \mathfrak{G}_0)$. By definition (5.28), it must be that $M_0 \supsetneq \mathfrak{G}_0$ and therefore, $M_0 = W_0$ by the maximality of \mathfrak{G}_0 . Thus M is an MGS of type (I) and thereby

$$M_1 = W'_1 \text{ or } W''_1. \quad (5.34)$$

Note that W''_1 is an irreducible \mathfrak{G}_0 -module, which follows from the irreducibility of \mathfrak{G}_0 and a simple fact that, as W_0 -modules,

$$W''_1 \cong (W_{-1})^*.$$

By our assumption, there is a $D \in \mathcal{M}_1(W_{-1}, \mathfrak{G}_0) \subset M_1$ with $\dim D \neq 0$. Assert that $D \notin W''_1$. Assuming on the contrary, by the irreducibility of W''_1 , we have $W''_1 \subset \mathcal{M}_1(W_{-1}, \mathfrak{G}_0)$ and thereby

$$W_0 = \text{alg}([W_{-1}, W''_1]) \subset \text{alg}([W_{-1}, \mathcal{M}_1(W_{-1}, \mathfrak{G}_0)]) \subset \mathfrak{G}_0.$$

This contradicts the assumption that \mathfrak{G}_0 is a nontrivial subalgebra of W_0 and hence the assertion holds. This proves that D belongs to neither W'_1 nor W''_1 , contradicting (5.34).

(b) For S , from Lemma 5.14, one implication is obvious. As in (a), we have $\mathcal{M}(S_{-1}, \mathfrak{G}_0)$ is maximal when $\mathcal{M}_1(S_{-1}, \mathfrak{G}_0) \neq 0$. For H , the conclusion follows from Lemmas 3.11(1) and 5.14.

(c) Suppose $L = K$. Assume on the contrary that $[1, u] = 0$ for every $u \in \mathcal{M}_1(K_{-1}, \mathfrak{G}_0)$. Then,

$$\mathcal{M}_1(K_{-1}, \mathfrak{G}_0) \subset K_{10}.$$

As in the proof of Proposition 5.10(2), we have $\mathcal{M}(K_{-1}, \mathfrak{G}_0)$ is not maximal.

Conversely, suppose $u = u_0 + u_1$ where $u_i \in K_{1i}$, $i = 0, 1$ and $u_1 \neq 0$. We claim that $u_0 \neq 0$. Indeed, by a direct computation, $[K_{-1}, K_{11}] = K_0$ holds. Assuming on the contrary that $u_0 = 0$, we have $u_1 \in \mathcal{M}(K_{-1}, \mathfrak{G}_0)$. K_{-1} is an irreducible \mathfrak{G}_0 -module, and so is K_{11} . Moreover, $K_{11} \subset \mathcal{M}(K_{-1}, \mathfrak{G}_0)$. Thus,

$$[K_{-1}, K_{11}] \subset [K_{-1}, \mathcal{M}(K_{-1}, \mathfrak{G}_0)] \subset \mathfrak{G}_0 \subsetneq K_0,$$

which contradicts $[K_{-1}, K_{11}] = K_0$.

Put $h \in K$, $h \notin \mathcal{M}(K_{-1}, \mathfrak{G}_0)$ and $\overline{M} = \text{alg}(\mathcal{M}(K_{-1}, \mathfrak{G}_0) + \mathbb{F}h)$. By definition (5.28) and the maximality of \mathfrak{G}_0 , we have $K_0 \subset \overline{M}$. Since the torus T is in \overline{M} , from Remark 2.3, there exists

$$v = v_0 + y_i z \in \text{alg}(\mathbb{F}u + T) \subset \overline{M}, \quad i \in \mathbf{I}, \quad v_0 \in K_{10}.$$

For $K_0 \subset \overline{M}$, without loss of generality, we may assume that $i \in \mathbf{I}_0$. If $v_0 = 0$, we have $y_i z \in \overline{M}$. For $u_0 \neq 0$, the conclusion holds. Otherwise, we claim that there exists a nonzero element in $\overline{M} \cap K_{10}$. Indeed, it is sufficient to consider the following cases.

Case 1. $D_{\tilde{i}}(v_0) \neq 0$. Note that $0 \neq [y^{(2\varepsilon_i)}, v] \in \overline{M} \cap K_{10}$.

Case 2. $D_{\tilde{i}}(v_0) = 0$ and there exists $t \in \mathbf{I}$, $t \neq i, \tilde{i}$, such that $D_{\tilde{t}}(v_0) \neq 0$. Note that $0 \neq [y_t y_i, v] \in \overline{M} \cap K_{10}$.

Case 3. $D_t(v_0) = 0$, for all $t \in \mathbf{I} \setminus \{i\}$. Then $v_0 = y^{(3\varepsilon_i)}$. Note that

$$y^{(3\varepsilon_i)} = (2\sigma(\tilde{i}))^{-1}([y_i y_{\tilde{i}}, v] - \sigma(\tilde{i})v) \in \overline{M} \cap K_{10}.$$

By Lemmas 2.1(2), 3.11(2) and $u_i \neq 0$ for $i = 0, 1$, the conclusion follows. \square

References

- [1] V. G. Kac. Lie superalgebras. *Adv. Math.* **26** (1977): 8–96.
- [2] M. Scheunert. Theory of Lie Superalgebras. *Lecture Notes in Math.* vol. **716**, Springer-Verlag, Berlin, 1979.
- [3] V. G. Kac. Classification of infinite dimensional simple linearly compact Lie superalgebras. *Adv. Math.* **139** (1998): 1–55.
- [4] Y.-Z. Zhang. Finite-dimensional Lie superalgebras of Cartan-type over fields of prime characteristic. *Chinese. Sci. Bull.* **42**(9) (1997): 720–724.
- [5] S. Bouarroudj and D. Leites. Simple Lie superalgebras and nonintegrable distributions in characteristic p . *J. Math. Sci.* **141**(4) (2007): 1390–1398.

- [6] A. Elduque. New simple Lie superalgebras in characteristic 3. *J. Algebra* **296**(1) (2006): 196–233.
- [7] E. B. Dynkin. Semisimple subalgebras of semisimple Lie algebras. *Mat. Sb. (N. S.)* **30**(72) (1952): 349–462; transl. *AMS Transl.* **6**(2) (1957): 111–244.
- [8] E. B. Dynkin. Maximal subgroups of the classical groups. *Trudy Moskov. Mat. Obsh.* **1** (1952): 39–166. transl. *AMS Transl.* **6**(2) (1957): 245–378.
- [9] G. M. Seitz. The maximal subgroups of classical algebraic groups. *Memories of the AMS* **67** (1987).
- [10] G. M. Seitz. Maximal subgroups of exceptional algebraic groups. *Memories of the AMS* **90** (1991).
- [11] M. Racine. On maximal subalgebras. *J. Algebra* **30**(1) (1974): 155–180.
- [12] M. Racine. Maximal subalgebras of exceptional Jordan algebras. *J. Algebra* **46** (1977): 12–21.
- [13] A. Elduque, J. Laliena, and S. Sacristan. Maximal subalgebras of associative superalgebras. *J. Algebra* **275**(1) (2004): 40–58.
- [14] A. Elduque, J. Laliena, and S. Sacristan. The Kac Jordan superalgebra: Automorphisms and maximal subalgebras. *Proc. AMS* **135**(12) (2007): 3805–3813.
- [15] Y. Barnea, A. Shalev and E. I. Zelmanov. Graded subalgebras of affine Kac-Moody algebras. *Israel J. Math.* **104** (1998): 321–334.
- [16] H. Melikyan. Maximal subalgebras of simple modular Lie algebras. *J. Algebra* **284** (2005): 824–856.
- [17] A. I. Kostrikin, I. R. Shafarevich. Graded Lie algebras of finite characteristic. *Izv. Akad. Nauk. SSSR Ser. Mat.* **33** (1969): 251–322 (in Russian); transl. *Math. USSR Izv.* **3** (1969): 237–304.
- [18] H. Strade. Simple Lie Algebras over Fields of Positive Characteristic I. Structure Theory. *de Gruyter Exp. Math.*, vol. **38**, Walter de Gruyter, Berlin, 2004.
- [19] H. Strade and R. Farnsteiner. Modular Lie Algebras and Their Representations. *Monographs and Textbooks in Pure Appl. Math.* vol **116**, Marcel Dekker, New York, 1988.
- [20] W.-D. Liu and Y.-Z. Zhang. Automorphism groups of restricted Cartan-type Lie superalgebras. *Comm. Algebra* **34**(10) (2006): 1–18.